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On the Ablowitz–Ladik equations with self-consistent sources

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Abstract

Darboux transformations and explicit solutions to Ablowitz–Ladik (AL) equations with self-consistent sources (ALESCS) are studied. Based on the Darboux transformation (DT) for the AL problem, we construct three types of non-auto-Bäcklund transformations connected with AL systems with different numbers of sources. The degenerate cases of DT and their applications to the reduced systems of ALESCS, for instance, discrete nonlinear Schrödinger with self-consistent sources (D-NLSSCS) and discrete mKdV equation with self-consistent sources (D-mKdVSCS), are discussed. Many types of solutions of ALESCS, D-NLSSCS and D-mKdVSCS, including solitons, positons, negatons can be derived from DTs and their degenerate cases.

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1. Introduction

Soliton equations with self-consistent sources (SESCS) have received much attention in recent years. It was found that these types of equations have important applications in many fields of physics. For example, the KdV equation with self-consistent sources represents the interaction of wave packets of high-frequency waves with a low-frequency wave, which is closely related to equations in plasma physics and hydrodynamics [1]. The KP equation with self-consistent sources describes interaction of a long wave with complex short wave packets propagating on the x–y plane [2, 3]. The nonlinear Schrödinger equation with self-consistent sources describes the interaction of the laser beam with a plasma [4, 5]. While mathematically, it turns out that SESCSs come out naturally from the *constrained flows* of soliton equations, which are essentially symmetry reductions of soliton systems. This point of view leads to not only the systematic constructions of SESCSs and their Lax representations [6–10], but also methods to construct explicit solutions. In the past few years, many SESCSs, including several 1+1, 2+1, dispersionless and discrete 1+1 systems have been constructed [11–17], various approaches for solving such system, for example, the inverse scattering method [4, 18–20], $\bar{\partial}$-method [21, 22], *Darboux transformations* (DT) [9, 14, 15, 23, 24], variable separating methods [25], bilinear Bäcklund transformations and Hirota bilinear method [26, 27], hodograph transformations [16, 17], have been used. New approach also appears in systematic construction of SESCSs. For example, very recently, Hu and his coworkers have succeeded in generating SESCSs from the Hirota bilinear method [28].

Ablowitz–Ladik (AL) lattice is a very important difference-differential system which was first introduced by M J Ablowitz and J F Ladik as the discrete counterpart of AKNS system [29, 30]. The AL lattice and its reductions, such as *discrete nonlinear Schrödinger equation* (D-NLS) and *discrete mKdV equation* (D-mKdV), have been studied intensively from both mathematical and physical points of view. Several methods have been used such as IST [30], Darboux transformations [31], Bäcklund transformations [32] etc, to discuss their solutions. However to the best of our knowledge, among many types of solutions, the positon and negaton solutions have not been studied yet.

Positons are singular solutions in contrast with solitons which were first investigated by V B Matveev [33]. For the KdV case, a positon solution is a slowly decreasing, oscillating solution and has the so-called supertransparent property. The positon solution was constructed by using the so-called generalized Darboux transformation. While negaton does not have supertransparency but shares the same idea of generalization, they differ by different choices of eigenvalues. From their expressions, it is easy to recognize since positon involves triangular functions and polynomials in x while negaton involves hyperbolic functions and polynomials in x.

In [13], the authors studied the constrained flows for AL as well as gave the discrete zero curvature representations of Ablowitz–Ladik equations with self-consistent sources (ALESCSs). However the solutions for ALESCSs or their reduced systems such as *D*-NLS with self-consistent sources (D-NLSSCS) and D-mKdV with self-consistent sources (D-mKdVSCS) have not been studied yet.

In this paper, we will study the solutions of the ALESCS as well as its reduced systems. Based on DTs for AL system [31, 32], we construct three types of non-auto-Bäcklund transformations among ALESCSs with different number of sources. They are named *modified* DTs (MDTs for short), which are roughly the variation of constants in ordinary DTs. They are much straightforward for solving SESCSs than the binary DTs with arbitrary function of time we used in [9, 23, 34]. We also deal with the multi-iteration formulae for MDTs and a generalized MDT. Based on careful analyses of cases of reductions, the D-NLSSCS and D-mKdVSCS are constructed and the generalized MDT for ALESCSs can be applied to D-NLSSCS and D-mKdVSCS. Then we show that many types of solutions, including especially positons and negatons can be constructed. As a by-product, solutions to original AL system can be obtained as solutions to ALESCS with zero sources. So actually we covered both problems we have mentioned in the last two paragraphs.

Our paper will be organized as follows. In section 2, we review briefly the construction of AL system with self-consistent sources. In section 3, we discuss three types of MDTs for ALESCSs and the multi-iteration formulae. In section 4, we give generalized MDT of type 1, which is called GMDT 1. In section 5, we reduce the ALESCSs to the D-NLSSCS and D-mKdVSCS and apply GMDT 1 to the reduced system. In section 6, we discuss various types of solutions to ALESCSs, D-NLSSCS and D-mKdVSCS. In section 7, we give conclusions and problems untouched in our paper.

2. The AL hierarchy with self-consistent sources

We give a schematic introduction to Ablowitz–Ladik hierarchy with self-consistent sources (ALHSCS). (See [13] for detail.)

2.1. The AL hierarchy

An AL equation (ALE) is given by

$$U_t = V^{(1)}U - UV, \qquad U = U(z,Q,R) := \begin{pmatrix} z & Q(n,t) \\ R(n,t) & 1/z \end{pmatrix}, \tag{1}$$

where for $f = f(n)$, $f^{(i)}$ means $f(n+i)$ $(n, i \in \mathbb{Z})$. V is a polynomial of z and z^{-1} with matrices coefficients depending on $Q^{(i)}$ and $R^{(i)}$ $(i \in \mathbb{Z})$ such that (1) is compatible. The Lax representation of (1) is

$$\psi^{(1)} = U\psi, \tag{2a}$$

$$\psi_t = V\psi, \tag{2b}$$

and the adjoint Lax representation is

$$\phi^{(-1)} = U^T\phi, \tag{3a}$$

$$-\phi_t = V^T\phi, \tag{3b}$$

where $\psi = (\psi^1(n,z,t), \psi^2(n,z,t))^T$ and $\phi = (\phi^1(n,z,t), \phi^2(n,z,t))^T$. (Hereafter we use superscripts ^{1}, ^{2} etc for components of vectors.)

The compatible V is constructed in the following way. Consider the solution of discrete stationary zero-curvature equation

$$\mathcal{V}^{(1)}U - U\mathcal{V} = 0,$$

if we take $\mathcal{V} = \mathcal{A} = \sum_{i=0}^{\infty} \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) z^{-i}$, then

$$\begin{aligned}
&\Delta a_0 = -\Delta d_0 = 0, \qquad b_{-1} = b_0 = 0, \qquad c_{-1} = c_0 = 0,\\
&\Delta a_{i+1} = -\Delta d_{i+1} = Qc_i - Rb_i^{(1)}, \qquad (i \geqslant 0),\\
&b_{i+1} = b_{i-1}^{(1)} + Q\big(a_i^{(1)} - d_i\big),\\
&c_{i+1}^{(1)} = c_{i-1} + R\big(a_i - d_i^{(1)}\big),
\end{aligned}$$

where $\Delta f = f^{(1)} - f$. If take $\mathcal{V} = \mathcal{B} = \sum_{i=0}^{\infty} \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) z^{i}$,

$$\begin{aligned}
&\Delta a_0 = -\Delta d_0 = 0, \qquad b_{-1} = b_0 = 0, \qquad c_{-1} = c_0 = 0,\\
&\Delta a_{i+1} = -\Delta d_{i+1} = Qc_i^{(1)} - Rb_i, \qquad (i \geqslant 0),\\
&b_{i+1}^{(1)} = b_{i-1} + Q\big(d_i - a_i^{(1)}\big),\\
&c_{i+1} = c_{i-1}^{(1)} + R\big(d_i^{(1)} - a_i\big).
\end{aligned}$$

It can be proved that the whole series of a_i, b_i, c_i, d_i can be expressed in polynomials of $Q^{(i)}$ and $R^{(i)}$ $(i \in \mathbb{Z})$ with some 'integral' constants in a_i and d_i. With out loss of generosity, we can assume $a_0 = 1$, $d_0 = -1$ and the other constants vanish for $i > 0$, i.e.

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2Q \\ 2R^{(-1)} & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} -2QR^{(-1)} & 0 \\ 0 & 2QR^{(-1)} \end{pmatrix} z^{-2} + \cdots,$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -2Q^{(-1)} \\ -2R & 0 \end{pmatrix} z^{1} + \begin{pmatrix} -2RQ^{(-1)} & 0 \\ 0 & 2RQ^{(-1)} \end{pmatrix} z^{2} + \cdots.$$

If we set

$$A_n := (z^{2n}\mathcal{A})_{\geqslant 0} + \delta_n, \qquad \delta_n = \begin{pmatrix} 0 & 0 \\ 0 & -d_{2n} \end{pmatrix} \qquad n \geqslant 0,$$

$$B_m := (z^{-2m}\mathcal{B})_{\leqslant 0} + \bar{\delta}_m, \qquad \bar{\delta}_m = \begin{pmatrix} -a_{2m} & 0 \\ 0 & 0 \end{pmatrix} \qquad m \geqslant 0.$$

Then the compatible V can be expressed as finite-term linear combinations

$$V = \sum x_n A_n + \sum y_m B_m \tag{4}$$

for constants x_n and y_m. We call the whole series of equations with such V the *AL hierarchy*.

Example 1 (some equations in AL hierarchy)**.**

- *The first nontrivial flow in AL hierarchy (ALFNF).*
  Let
  $$V = V_1 := A_1, \tag{5}$$
  then (1) reads
  $$Q_t = 2(1 - QR)Q^{(1)}, \qquad R_t = -2(1 - QR)R^{(-1)}, \tag{6}$$
  which we call as ALFNF.
- *Discrete nonlinear Schrödinger equation (D-NLS).*
  Let
  $$V = V_2 := \mathrm{i}\left(A_0 + B_0 - \tfrac{1}{2}A_1 - \tfrac{1}{2}B_1\right), \tag{7}$$
  under restrictions $R = \pm Q^*$ where * is the complex conjugation, (1) becomes
  $$\mathrm{i}Q_t = Q^{(1)} + Q^{(-1)} - 2Q \mp 2|Q|^2(Q^{(1)} + Q^{(-1)}), \tag{8}$$
  which are the discrete version of nonlinear Schrödinger equations introduced in [29, 30].
- *Discrete modified KdV equation (D-mKdV).*
  Let
  $$V = V_3 := \tfrac{1}{2}(A_1 - B_1), \tag{9}$$
  under restrictions $R = \pm Q$, (1) becomes
  $$Q_t = (1 \mp Q^2)(Q^{(1)} - Q^{(-1)}), \tag{10}$$
  which are the discrete mKdV equations introduced in [29, 30].

2.2. General scheme for AL Hierarchy with self-consistent sources

The Hamiltonian formalism for (1) is

$$\begin{pmatrix} Q_t \\ R_t \end{pmatrix} = (1 - QR)K \begin{pmatrix} \delta H/\delta Q \\ \delta H/\delta R \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for corresponding Hamiltonian H (see [13]). Variation derivative of spectral parameter z holds

$$\left(\frac{\delta z}{\delta Q}, \frac{\delta z}{\delta R}\right) = (\psi^2\phi^1, \psi^1\phi^2),$$

where ψ and ϕ satisfy (2) and (3), respectively. We define *constrained flows* as a discrete system for variables Q, R, ψ_i and ϕ_i

$$\frac{\delta H}{\delta Q} = \sum_{i=1}^{N} \frac{\delta z_i}{\delta Q}, \qquad \frac{\delta H}{\delta R} = \sum_{i=1}^{N} \frac{\delta z_i}{\delta R},$$

$$\psi_i^{(1)} = U(z_i)\psi_i, \qquad \phi_i^{(-1)} = U(z_i)^T\phi_i, \qquad i = 1, \ldots, N.$$

Then the AL hierarchy with N self-consistent sources (ALHSCS) are differential-difference systems which take above-constrained flows as their stationary systems:

$$\begin{pmatrix} Q_t \\ R_t \end{pmatrix} = (1-QR)K\begin{pmatrix} \delta H/\delta Q - \sum \delta z_i/\delta Q \\ \delta H/\delta R - \sum \delta z_i/\delta R \end{pmatrix}, \tag{11a}$$

$$\psi_i^{(1)} = U(z_i)\psi_i, \qquad \phi_i^{(-1)} = U(z_i)^T\phi_i, \qquad i=1,\ldots,N. \tag{11b}$$

The Lax representation for (11a) (under (11b)) is given by

$$\psi^{(1)} = U(z,Q,R)\psi, \tag{12a}$$

$$\psi_t = \left(V(z,Q,R) + \sum_{i=1}^{N} X_i\right)\psi, \tag{12b}$$

where

$$X_i = X(z,z_i,\psi_i,\phi_i) = \frac{1}{z^2-z_i^2}\left[S_i\psi_i\cdot\left(S_i\phi_i^{(-1)}\right)^T + \psi_i^T\phi_i^{(-1)}\cdot T_i\right],$$

$$S_i = \mathrm{diag}(z_i,z),\, T_i = \mathrm{diag}\left(\frac{z^2-3z_i^2}{4}, \frac{z_i^2-3z^2}{4}\right).$$

Example 2 (the ALFNF with self-consistent sources (ALFNFSCS))**.** Let $V = V_1$ (see (5)), then the compatibility of (12) gives ALFNFSCS

$$Q_t = (1-QR)\left(2Q^{(1)} - \sum_{i=1}^{N}\psi_i^1\phi_i^2\right), \qquad R_t = -(1-QR)\left(2R^{(-1)} - \sum_{i=1}^{N}\psi_i^2\phi_i^1\right), \tag{13a}$$

$$\psi_i^{(1)} = U(z_i)\psi_i, \qquad \phi_i^{(-1)} = U(z_i)^T\phi_i, \qquad i=1,\ldots,N. \tag{13b}$$

3. Darboux transformations for some equations in ALHSCSs

3.1. Darboux transformations for A-L isospectral problem

Based on [31] and [32], we can construct three types of DTs for AL isospectral problem.

Let $\mathcal{D} = \mathcal{D}(z)$ be a 2×2 matrix such that $\tilde{U} = \mathcal{D}^{(1)}U\mathcal{D}^{-1}$ has the same form as U. The simplest Darboux matrix $\mathcal{D}$ has the form

$$\mathcal{D}_1 := \begin{pmatrix} z+vz^{-1} & \sigma \\ \tau & uz+z^{-1} \end{pmatrix} \tag{14}$$

where σ, τ, u, v, Q, R and new potentials $\tilde{Q}, \tilde{R}$ in $\tilde{U}$ satisfy

$$\begin{aligned} &\tilde{Q} = \sigma^{(1)} + v^{(1)}Q, && \tilde{R} = \tau^{(1)} + u^{(1)}R, \\ &Q = \sigma + u\tilde{Q}, && R = \tau + v\tilde{R}, \\ &v^{(1)} + \sigma^{(1)}R = v + \tau\tilde{Q}, && u^{(1)} + \tau^{(1)}Q = u + \sigma\tilde{R}. \end{aligned}$$

$\mathcal{D}_1$ has two consistent reductions: if we impose $u \equiv 0$ and $\sigma \equiv Q$ in $\mathcal{D}_1$, we find Darboux matrix

$$\mathcal{D}_2 := \begin{pmatrix} z+vz^{-1} & Q \\ \tau & z^{-1} \end{pmatrix}, \tag{15}$$

where τ, v, Q, R and new potentials $\tilde{Q}$, $\tilde{R}$ satisfy

$$\begin{aligned} &\tilde{Q} = Q^{(1)} + v^{(1)}Q, && \tilde{R} = \tau^{(1)}, \\ &R = \tau + v\tilde{R}, && v^{(1)} + Q^{(1)}R = v + \tau\tilde{Q}. \end{aligned}$$

If we impose $v \equiv 0$ and $\tau \equiv R$ in $\mathcal{D}_1$, we find another Darboux matrix

$$\mathcal{D}_3 := \begin{pmatrix} z & \sigma \\ R & uz + z^{-1} \end{pmatrix}, \tag{16}$$

where σ, u, Q, R and new potentials $\tilde{Q}$, $\tilde{R}$ satisfy

$$\begin{aligned} &\tilde{Q} = \sigma^{(1)}, && \tilde{R} = \tau^{(1)} + u^{(1)}R, \\ &Q = \sigma + u\tilde{Q}, && u^{(1)} + \tau^{(1)}Q = u + \sigma\tilde{R}. \end{aligned}$$

We show the Darboux transformations for (2) as follows.

Proposition 1. *If $h_j = \psi(\zeta_j)$ are eigenfunctions for Lax pair (2) with $z = \zeta_j$ $(j = 1, 2)$, (hereafter we restrict our attention to $V = V_m, m = 1, 2, 3$, see example 1) , then $\tilde{U} = U(\tilde{Q}, \tilde{R}, z)$, $\tilde{V} = V_m(\tilde{Q}, \tilde{R}, z)$ and $\tilde{\psi} = \mathcal{D}_i\psi$ give new solutions to (2). Hence $\tilde{Q}$, $\tilde{R}$ are new solutions to (1). Here are the detailed Darboux transformations:*

- *DT 1:*

$$\begin{aligned} &\tilde{Q} = \sigma^{(1)} + v^{(1)}Q, \qquad \tilde{R} = \tau^{(1)} + u^{(1)}R, \\ &\sigma := -\frac{\begin{vmatrix} h_1^1/\zeta_1 & h_1^1\zeta_1 \\ h_2^1/\zeta_2 & h_2^1\zeta_2 \end{vmatrix}}{\begin{vmatrix} h_1^1/\zeta_1 & h_1^2 \\ h_2^1/\zeta_2 & h_2^2 \end{vmatrix}}, \qquad \tau := -\frac{\begin{vmatrix} h_1^2/\zeta_1 & h_1^2\zeta_1 \\ h_2^2/\zeta_2 & h_2^2\zeta_2 \end{vmatrix}}{\begin{vmatrix} h_1^1 & h_1^2\zeta_1 \\ h_2^1 & h_2^2\zeta_2 \end{vmatrix}}, \\ &u := -\frac{\begin{vmatrix} h_1^1 & h_1^2/\zeta_1 \\ h_2^1 & h_2^2/\zeta_2 \end{vmatrix}}{\begin{vmatrix} h_1^1 & h_1^2\zeta_1 \\ h_2^1 & h_2^2\zeta_2 \end{vmatrix}}, \qquad v := -\frac{\begin{vmatrix} h_1^1\zeta_1 & h_1^2 \\ h_2^1\zeta_2 & h_2^2 \end{vmatrix}}{\begin{vmatrix} h_1^1/\zeta_1 & h_1^2 \\ h_2^1/\zeta_2 & h_2^2 \end{vmatrix}}, \\ &\tilde{\psi} := \mathcal{D}_1\psi = \left[\frac{\begin{vmatrix} \psi^1/z & \psi^2 & z\psi^1 \\ h_1^1/\zeta_1 & h_1^2 & \zeta_1 h_1^1 \\ h_2^1/\zeta_2 & h_2^2 & \zeta_2 h_2^1 \end{vmatrix}}{\begin{vmatrix} h_1^1/\zeta_1 & h_1^2 \\ h_2^1/\zeta_2 & h_2^2 \end{vmatrix}}, \frac{\begin{vmatrix} \psi^1 & \psi^2 z & \psi^2/z \\ h_1^1 & h_1^2\zeta_1 & h_1^2/\zeta_1 \\ h_2^1 & h_2^2\zeta_2 & h_2^2/\zeta_2 \end{vmatrix}}{\begin{vmatrix} h_1^1 & h_1^2\zeta_1 \\ h_2^1 & h_2^2\zeta_2 \end{vmatrix}} \right]^T. \end{aligned} \tag{17}$$

- *DT 2:*

$$\begin{aligned} &\tilde{Q} := Q^{(1)} + v^{(1)}Q, \qquad \tilde{R} := \tau^{(1)}, \\ &\tau := -\frac{h_1^2}{\zeta_1 h_1^1}, \qquad v := -\zeta_1^2 - \frac{\zeta_1 h_1^2}{h_1^1}Q, \\ &\tilde{\psi} = \mathcal{D}_2\psi = \left[\frac{\begin{vmatrix} \psi^{1(1)} & h_1^{1(1)} \\ \psi^1/z & h_1^1/\zeta_1 \end{vmatrix}}{h_1^1/\zeta_1}, \frac{\begin{vmatrix} \psi^{2(1)} & h_1^{2(1)} \\ \psi^1 & h_1^1 \end{vmatrix}}{h_1^1} \right]^T. \end{aligned}$$

- *DT 3:*

$$\tilde{Q} := \sigma^{(1)}, \qquad \tilde{R} := R^{(1)} + u^{(1)} R,$$

$$\sigma := -\frac{h_1^1 \zeta_1}{h_1^2}, \qquad u := -\zeta_1^{-2} - \frac{h_1^1}{\zeta_1 h_1^2} R,$$

$$\tilde{\psi} = \mathcal{D}_3 \psi = \left[\frac{\begin{vmatrix} \psi^{1(1)} & h_1^{1(1)} \\ \psi^2 & h_1^2 \end{vmatrix}}{h_1^2}, \frac{\begin{vmatrix} \psi^{2(1)} & h_1^{2(1)} \\ \psi^2 z & h_1^2 \zeta_1 \end{vmatrix}}{h_1^2 \zeta_1} \right]^T .$$

Remark 1. In [31], the author considered Darboux matrix for the Ablowitz–Ladik isospectral problem with four potentials which is essentially equivalent to the first Darboux matrix in our case. In [32], Bäcklund transformations were studied for Ablowitz–Ladik hierarchy with two potentials, the corresponding transformation matrices they studied are similar to the second and third Darboux matrices, whereas the explicit form of DT of the second and third types are missed in the literature.

Proof. The proof of DT 1 can be found in [31]. DT 2, DT 3 can be proved by straightforward calculations. □

3.2. Darboux transformations for equations in ALHSCS

Proposition 2 (Darboux transformations for ALESCS)**.** *Suppose* $h_j := \psi(n, t, \zeta_j)$ $(j = 1, 2)$ *satisfy Lax pair (12) with* $z = \zeta_j$, $V = V_m$ $(m = 1, 2, 3)$. *If we define* $A^{\perp} := ((A^{-1})^T)^{(1)}$ *for* 2×2 *matrix* A, *then* $\tilde{Q}$, $\tilde{R}$ *and* $\tilde{\psi}$ *determined by DT* k $(k = 1, 2, 3)$ *as well as*

$$\tilde{\psi}_i := \mathcal{D}_k(z_i)\psi_i, \qquad \tilde{\phi}_i := \mathcal{D}_k^{\perp}(z_i)\phi_i, \qquad i = 1, \ldots, N$$

satisfy again (12) and (11b). Hence we get a new solution to (11).

Proof. By straightforward calculation. □

Remark 2. We should mention that it is difficult to obtain non-trivial solutions to ALESCS from $Q = R = 0$, $\psi_i = \phi_i = 0$ by proposition 2.

So we need 'modified' DT 1, 2, 3 which is stated in the next subsection.

3.3. Non-auto-Bäcklund transformations between ALE with different numbers of sources

Suppose there are two pairs of eigenfunctions of (12): $f_j := f(n, t, \zeta_j)$ and $g_j := g(n, t, \zeta_j)$ $(j = 1, 2)$, then proposition 2 still holds if we make use of $h_j = f_j + \alpha_j g_j$ (α_j is a constant) in it. Since the expressions of DT 1 (DT 2, DT 3) do not involve partial derivatives about t, it is easy to see if α_j is substituted by function $\alpha_j(t)$, the corresponding new variables satisfy (12*a*) and (11*b*) as well. For (12*b*), new terms containing $\frac{d\alpha_j}{dt}$ will appear on the left-hand side. Thus new terms must be added on the right-hand side to make it still an equality. The new adding term can be well expressed in terms of new self-consistent sources and we get (12) with new variables and $N + 2$ ($N + 1$, respectively) self-consistent sources. Hereafter we call such modifications of DT 1 (DT 2, DT 3) with arbitrary function of time the MDT 1 (MDT 2, MDT 3).

Theorem 1 (MDT 1 (MDT 2, MDT 3) for ALESCS)**.** *Suppose* $f_j := f(n,t,\zeta_j), g_j := g(n,t,\zeta_j)$ *are independent eigenfunctions for (12),* $\alpha_j(t)$ *are arbitrary functions of time for* $j = 1, 2$ *(*$j = 1$ *respectively). Applying DT 1 (DT 2, DT 3 respectively) with* $h_j := f_j + \alpha_j(t) g_j$*, we get new solution to (12) with N increases to* $N+2$ *(*$N+1$ *respectively), namely* $\tilde{Q}, \tilde{R}, \tilde{\psi}, \tilde{\psi}_i, \tilde{\phi}_i$ $(i = 1, \ldots, N)$ *and the following* $\tilde{\psi}_{N+j}, \tilde{\phi}_{N+j}$ *with* $z_{N+j} = \zeta_j$ *(*$j = 1, 2$ *(or* $j = 1$*)) together satisfy (12), (11b) and hence (11a).*

- *For DT 1:*

$$\tilde{\psi}_{N+j} = \mathcal{D}_1(\zeta_j) f_j,$$

$$\tilde{\phi}_{N+j} = \frac{\dot{\alpha}_j}{\alpha_j}\left[\frac{z^2-\zeta_j^2}{z^2}\mathcal{D}_1^{\perp}(z)\right]\Bigg|_{z=\zeta_j} K\left(\frac{g_j}{\det(f_j, g_j)}\right)^{(1)}, \qquad j = 1, 2.$$

- *For DT 2:*

$$\tilde{\psi}_{N+1} = \mathcal{D}_2(\zeta_1) f_1,$$

$$\tilde{\phi}_{N+1} = \frac{\dot{\alpha}_1}{\alpha_1}\left[\frac{z^2-\zeta_1^2}{z^2}\mathcal{D}_2^{\perp}(z)\right]\Bigg|_{z=\zeta_1} K\left(\frac{g_1}{\det(f_1, g_1)}\right)^{(1)}.$$

- *For DT 3:*

$$\tilde{\psi}_{N+1} = \mathcal{D}_3(\zeta_1) f_1,$$

$$\tilde{\phi}_{N+1} = \frac{\dot{\alpha}_1}{\alpha_1}\left[\frac{z^2-\zeta_1^2}{z^2}\mathcal{D}_3^{\perp}(z)\right]\Bigg|_{z=\zeta_1} K\left(\frac{g_1}{\det(f_1, g_1)}\right)^{(1)},$$

where $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. By straightforward calculation. □

Remark 3. The 'variation of constant' is a well-known method for solving non-homogeneous linear ODEs. Here we apply 'variation of constant' to Darboux transformations with resemblance that varying α_j to $\alpha_j(t)$ will add new sources (non-homogeneous terms) to the corresponding nonlinear differential-difference system.

Theorem 1 establishes non-auto-Bäcklund transformations from ALESCSs with N to $N+2$ ($N+1$ respectively) self-consistent sources.

3.4. Formulae of multi-iteration for MDT of types 1 and 2 and 3

For l-time iteration of MDT 1, to simplify the expression we define some notations. Suppose $h = (h^1(\zeta), h^2(\zeta))^T$, define $2l$-dimensional vectors

$$A_{2l}(h,\zeta) := (h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ \cdots\ h^1\zeta^{l-2},\ h^2\zeta^{1-l},\ h^2\zeta^{3-l},\ \cdots\ \cdots\ h^2\zeta^{l-1})^T,$$
$$B_{2l}(h,\zeta) := (h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ \cdots\ \cdots\ h^1\zeta^{l},\ h^2\zeta^{3-l},\ h^2\zeta^{5-l},\ \cdots\ h^2\zeta^{l-1})^T,$$
$$C_{2l}(h,\zeta) := (h^1\zeta^{l},\ h^1\zeta^{2-l},\ h^1\zeta^{4-l},\ \cdots\ h^1\zeta^{l-2},\ h^2\zeta^{1-l},\ h^2\zeta^{3-l},\ \cdots\ \cdots\ h^2\zeta^{l-1})^T,$$
$$D_{2l}(h,\zeta) := (h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ h^1\zeta^{l-4},\ h^2\zeta^{-l-1},\ h^2\zeta^{1-l},\ \cdots\ \cdots\ \cdots\ h^2\zeta^{l-1})^T,$$
$$E_{2l}(h,\zeta) := (h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ \cdots\ h^1\zeta^{l-2},\ h^2\zeta^{1-l},\ h^2\zeta^{3-l},\ \cdots\ h^2\zeta^{l-3},\ h^2\zeta^{-l-1})^T,$$

and $(2l+1)$-dimensional vectors

$$X_{2l+1}(h,\zeta) := (h^1\zeta^{l},\ h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ h^1\zeta^{l-2},\ h^2\zeta^{1-l},\ h^2\zeta^{3-l},\ \cdots\ h^2\zeta^{l-1})^T,$$
$$Y_{2l+1}(h,\zeta) := (h^2\zeta^{-1-l},\ h^1\zeta^{-l},\ h^1\zeta^{2-l},\ \cdots\ h^1\zeta^{l-2},\ h^2\zeta^{1-l},\ h^2\zeta^{3-l},\ \cdots\ h^2\zeta^{l-1})^T.$$

For vectors $h_j = \left(h_j^1(\zeta_j), h_j^2(\zeta_j)\right)^T$ for $j = 1, \ldots, 2l$ and $\psi = (\psi^1(z), \psi^2(z))^T$, define determinants

$$A[l] := \det(A_{2l}(h_1, \zeta_1), \ldots, A_{2l}(h_{2l}, \zeta_{2l})),$$

and $B[l], \ldots, E[l]$ are defined by substituting letter A by $B, \ldots, E$.

$$X[l](\psi) := \det(X_{2l+1}(\psi, z), X_{2l+1}(h_1, \zeta_1), \ldots, X_{2l+1}(h_{2l}, \zeta_{2l})),$$

and $Y[l](\psi)$ is defined by substituting X by Y.

Theorem 2 (multi-iteration of MDT 1). *Let f_j, g_j be independent eigenfunctions to (12) (for $V = V_k, k = 1, 2, 3$) with $z = \zeta_j$, ($j = 1, \ldots, 2l$). Let $\alpha_j(t)$ be arbitrary functions, $h_j = f_j + \alpha_j g_j$. Then l-iteration of MDT 1 reads*

$$Q[l] = -\frac{B[l]^{(1)}}{A[l]^{(1)}} - Q\frac{C[l]^{(1)}}{A[l]^{(1)}}, \qquad R[l] = -\frac{D[l]^{(1)}}{A[l]^{(1)}} - R\frac{E[l]^{(1)}}{A[l]^{(1)}}, \tag{18a}$$

$$\psi_i[l] = \left(\frac{X[l](\psi_i)}{A[l]}, \frac{z_i Y[l](\psi_i)}{A[l]}\right)^T, \qquad i = 1, \ldots, N, \tag{18b}$$

$$\phi_i[l] = \frac{z_i^{2l}}{\prod_{j=1}^{2l}\left(z_i^2 - \zeta_j^2\right)}\left(\frac{z_i Y[l](K\phi_i^{(-1)})}{E[l]}, -\frac{X[l](K\phi_i^{(-1)})}{E[l]}\right)^{(1)T}, \tag{18c}$$

$$\psi_{N+j}[l] = \left(\frac{X[l](f_j)}{A[l]}, \frac{\zeta_j Y[l](f_j)}{A[l]}\right)^T, \qquad j = 1, \ldots, 2l, \tag{18d}$$

$$\phi_{N+j}[l] = -\frac{\zeta_j^{2l-2}\dot{\alpha}_j/\alpha_j}{\det(f_j, g_j)^{(1)}\prod_{1\leqslant i\neq j\leqslant 2l}\left(\zeta_j^2 - \zeta_i^2\right)}\left(\frac{\zeta_j Y[l](g_j)}{E[l]}, -\frac{X[l](g_j)}{E[l]}\right)^{(1)T}, \tag{18e}$$

(18) gives a new solution to (11) with $Q[l], R[l]$ and $\psi_i[l], \phi_i[l]$ corresponding to z_i ($i = 1, \ldots, N$), new sources $\psi_{N+j}[l], \phi_{N+j}[l]$ corresponding to $z_{N+j} = \zeta_j$ ($j = 1, \ldots, 2l$).

Proof. The formula for multi-iteration of DT for 4-potential A-L equations was proved in [31]. Here we describe briefly how to give proof in our case (with source terms).

The l-iteration makes use of l pairs of vectors $\{h_{2j-1}, h_{2j}\}$. It is easy to see if the jth Darboux matrix is denoted by $\mathcal{D}_1[j-1] : U[j-1] \mapsto U[j]$, then

$$\begin{aligned} U[j] &= \mathcal{D}_1[j-1]^{(1)}U[j-1]\mathcal{D}_1[j-1]^{-1} \\ &= (\mathcal{D}_1[j-1]\mathcal{D}_1[j-2]\cdots\mathcal{D}_1)^{(1)}U(\mathcal{D}_1[j-1]\mathcal{D}_1[j-2]\cdots\mathcal{D}_1)^{-1}. \end{aligned}$$

Denoting by $\mathcal{D}_1(l) = \mathcal{D}_1[l-1]\mathcal{D}_1[l-2]\cdots\mathcal{D}_1$, $\mathcal{D}_1(l)$ can be written as

$$\mathcal{D}_1(l) = \begin{pmatrix} z^l + \sum_{j=0}^{l-1} v_{2j-l}z^{2j-l} & \sum_{j=0}^{l-1}\sigma_{2j+1-l}z^{2j+1-l} \\ \sum_{j=0}^{l-1}\tau_{2j+1-l}z^{2j+1-l} & \sum_{j=0}^{l-1}u_{2j+2-l}z^{2j+2-l} + z^{-l} \end{pmatrix} \tag{19}$$

with coefficients $v_{2j-l}, \sigma_{2j+1-l}, \tau_{2j+1-l}, u_{2j+2-l}$ to be determined later. By virtue of Darboux transformation, it is easy to see $\mathcal{D}_1(l)|_{z=\zeta_j}h_j = 0$ for $j = 1, 2, \ldots, 2l$. So the coefficients satisfy the following linear equation system:

$$\begin{pmatrix} A_{2l}(h_1,\zeta_1)^T \\ A_{2l}(h_2,\zeta_2)^T \\ \vdots \\ A_{2l}(h_{2l},\zeta_{2l})^T \end{pmatrix} \begin{pmatrix} v_{-l} \\ v_{2-l} \\ \vdots \\ v_{l-2} \\ \sigma_{1-l} \\ \sigma_{3-l} \\ \vdots \\ \sigma_{l-1} \end{pmatrix} = -\begin{pmatrix} \zeta_1^l h_1^1 \\ \vdots \\ \vdots \\ \zeta_{2l}^l h_{2l}^1 \end{pmatrix}, \qquad \begin{pmatrix} A_{2l}(h_1,\zeta_1)^T \\ A_{2l}(h_2,\zeta_2)^T \\ \vdots \\ A_{2l}(h_{2l},\zeta_{2l})^T \end{pmatrix} \begin{pmatrix} \tau_{1-l} \\ \tau_{3-l} \\ \vdots \\ \tau_{l-1} \\ u_{2-l} \\ u_{4-l} \\ \vdots \\ u_l \end{pmatrix} = -\begin{pmatrix} \zeta_1^{-l-1} h_1^2 \\ \vdots \\ \vdots \\ \zeta_{2l}^{-l-1} h_{2l}^2 \end{pmatrix},$$

and can be solved by the Cramer rule. By comparing the coefficients of z^n and z^{-n} in the equation $\mathcal{D}_1(l)^{(1)}U = U[l]\mathcal{D}_1(l)$ we find (18a). The l-iteration formulae for arbitrary eigenfunction ψ_i w.r.t. eigenvalue z_i is $\psi_i[l] = \mathcal{D}_1(l)|_{z=z_i}\psi_i$, which is (18$b$) by making use of Laplace expansion of determinant. Formulae for arbitrary adjoint eigenfunction w.r.t. z_i is $\phi_i[l] = \mathcal{D}_1(l)^{\perp}|_{z=z_i}\phi_i$. Since $\mathcal{D}_1(l)^{\perp} = ((\mathcal{D}_1(l)^T)^{(1)})^{-1} = ((\mathcal{D}_1(l)^T)^*)^{(1)}/\det\mathcal{D}_1(l)^{(1)}$, we have to calculate $\det\mathcal{D}_1(l)$. Note that each $\det\mathcal{D}_1[j] = a_0(z^4 + a_1z^2 + a_2)/z^2$ for some coefficients a_0, a_1 and a_2. Then according to the definition of $\mathcal{D}_1(l)$ and (19), we have

$$\det\mathcal{D}_1(l) = u_l P(z^2)/z^{2l}, \qquad \deg P(z) = 2l.$$

Because ζ_j, $j = 1,\ldots,2l$ are zeros for $\det\mathcal{D}_1(l)$, it is easy to see $P(z^2) = \prod_{i=1}^{2l}(z^2-\zeta_i^2)$. Thus (18$c$) can be obtained by Laplace expansion analogously. Formulae (18d) and (18e) are proved by noticing that f_j and $K(g_j/\det(f_j,g_j))^{(1)}$ are eigenfunctions and adjoint eigenfunctions w.r.t. eigenvalue ζ_j. □

For l-iteration of MDT 2, define

$$A_{l+1}(h,\zeta) := \left(h^{1^{(l)}},\quad h^{1^{(l-1)}}/\zeta,\quad h^{1^{(l-2)}}/\zeta^2,\quad \ldots,\quad h^1/\zeta^l\right)^T,$$
$$B_{l+1}(h,\zeta) := \left(h^{2^{(l)}},\quad h^{1^{(l-1)}},\quad h^{1^{(l-2)}}/\zeta,\quad \ldots,\quad h^1/\zeta^{l-1}\right)^T,$$
$$\Lambda_{l+1} := \operatorname{diag}\left(\prod_{j=0}^{l-1}(1-QR)^{(j)}, \prod_{j=0}^{l-2}(1-QR)^{(j)}, \ldots, (1-QR), 1\right).$$

Let

$$A[l] := \det(A_l(h_1,\zeta_1),\ldots,A_l(h_l,\zeta_l)),$$
$$A[l](\psi) := \det(A_{l+1}(\psi,z), A_{l+1}(h_1,\zeta_1),\ldots,A_{l+1}(h_l,\zeta_l)),$$
$$\tilde{A}[l](\psi) := \det(\Lambda_{l+1}A_{l+1}(\psi,z), A_{l+1}(h_1,\zeta_1),\ldots,A_{l+1}(h_l,\zeta_l)),$$

and $B[l]$, $B[l](\psi)$ and $\tilde{B}[l](\psi)$ are defined by substituting 'A' by 'B'.

Theorem 3 (l-iteration of MDT 2)**.** *Let f_j, g_j be independent eigenfunctions to (12) with $z = \zeta_j$ for $j = 1,\ldots,l$. Let $\alpha_j(t)$ be arbitrary functions, $h_j = f_j + \alpha_j g_j$. Then formulae*

$$Q[l] = -\prod_{i=1}^{l}\zeta_i \times \frac{A[l]^{(1)}}{B[l]^{(1)}} + (1-QR)\prod_{i=1}^{l}\zeta_i \times \frac{A[l]A[l]^{(2)}}{A[l]^{(1)}B[l]^{(1)}}, \qquad (20a)$$

$$R[l] = -\frac{B[l]^{(1)}}{A[l]^{(1)}}\bigg/\prod_{i=1}^{l}\zeta_i, \qquad (20b)$$

$$\psi_i[l]=\left(\frac{A[l](\psi_i)\prod_{s=1}^{l}\zeta_s}{A[l]},\frac{B[l](\psi_i)}{A[l]}\right)^T,\qquad i=1,\ldots,N \tag{20c}$$

$$\phi_i[l]=-\frac{z_i^{2l}\prod_{j=1}^{l}\zeta_j}{\prod_{j=1}^{l}\left(z_i^2-\zeta_j^2\right)}\left(\frac{\tilde{B}[l]\left(K\phi_i^{(-1)}\right)}{A[l]^{(1)}},-\frac{\tilde{A}[l]\left(K\phi_i^{(-1)}\right)}{A[l]^{(1)}}\prod_{j=1}^{l}\zeta_j\right)^{(1)T}, \tag{20d}$$

$$\psi_{N+j}[l]=\left(\frac{A[l](f_j)\prod_{s=1}^{l}\zeta_s}{A[l]},\frac{B[l](f_j)}{A[l]}\right)^T,\qquad j=1,\ldots,l \tag{20e}$$

$$\phi_{N+j}[l]=\frac{\dot{\alpha}_j/\alpha_j\cdot\zeta_j^{2l-2}\prod_{s=1}^{l}\zeta_s}{\det(f_j,g_j)^{(1)}\prod_{1\leqslant i\neq j\leqslant l}\left(\zeta_j^2-\zeta_i^2\right)}\left(\frac{B[l](g_j)}{A[l]^{(1)}},-\frac{A[l](g_j)}{A[l]^{(1)}}\prod_{s=1}^{l}\zeta_s\right)^{(1)T}, \tag{20f}$$

give a new solution to (11) with $Q[l]$, $R[l]$ *and* $\psi_i[l]$, $\phi_i[l]$ *corresponding to* z_i $(i=1,\ldots,N)$, *new sources* $\psi_{N+j}[l]$, $\phi_{N+j}[l]$ *corresponding to* $z_{N+j}=\zeta_j$ $(j=1,\ldots,l)$.

Proof. Due to the obscurity of l-iteration Darboux matrix $\mathcal{D}_2(l)$, similar argument as MDT 1 cannot be applied to this case. We use induction. Firstly if $(l-1)$-iteration of ψ_i and $h_l[l-1]$ satisfy (20c) then it can be proved that l-iteration of eigenfunction ψ_i satisfies (20c).

Let $\mathcal{D}_2[l-1]$ be the Darboux matrix (15) with coefficient $v[l-1]$, $Q[l-1]$ and $\tau[l-1]$. Set $h_l[l-1]$ to be the l-iteration of eigenfunction h_l w.r.t. ζ_l. Since $\mathcal{D}_2[l-1]|_{z=\zeta_l}h_l[l-1]=0$, we have $\tau[l-1]=-\frac{h_l^2[l-1]}{h_l^1[l-1]\zeta_l}$. Then $R[l]=\tau[l-1]^{(1)}$ which is (20b).

$Q[l]$ is obtained by the aid of $\det U[l]=1-Q[l]R[l]$. Since $U[l]=(\mathcal{D}_2[l-1]\cdots\mathcal{D}_2[0])^{(1)}U(\mathcal{D}_2[l-1]\cdots\mathcal{D}_2[0])^{-1}$, to determine $\det U[l]$, we must determine $\det\mathcal{D}_2[j]$ first. By virtue of (15) and the fact $\det\mathcal{D}_2[j]|_{z=\zeta_{j+1}}=0$ we have $\det\mathcal{D}_2[j]=(1-Q[j]\tau[j])\left(z^2-\zeta_{j+1}^2\right)/z^2$. Inserting the expression $\tau[j]=-\frac{h_{j+1}^2[j]}{h_{j+1}^1[j]\zeta_{j+1}}$ into $\det\mathcal{D}_2[j]$, recalling that $h_{j+1}[j]$ satisfies $h_{j+1}[j]^{(1)}=U[j]|_{z=\zeta_{j+1}}h_{j+1}[j]$, we have

$$\det\mathcal{D}_2[j]=\left(z^2-\zeta_{j+1}^2\right)/z^2\,\frac{h_{j+1}^1[j]^{(1)}}{\zeta_{j+1}h_{j+1}^1[j]}=\left(z^2-\zeta_{j+1}^2\right)/z^2\,\frac{A[j+1]^{(1)}A[j]}{A[j]^{(1)}A[j+1]\zeta_{j+1}},$$

and trivially $\det\mathcal{D}_2[0]=\left(z^2-\zeta_1^2\right)/z^2\,\frac{(h_1^1)^{(1)}}{\zeta_1h_1^1}$. Substituting all into $\det U[l]$ it is easy to find

$$\det U[l]=1-Q[l]R[l]=(1-QR)\frac{A[l]^{(2)}A[l]}{(A[l]^{(1)})^2}.$$

Then (20a) obtains.

The l-iteration of adjoint eigenfunction (20d) can be worked out as follows.

Suppose Φ is the fundamental solution (2×2 matrix) to (3a) w.r.t. z, then $\Phi^\sharp:=((\Phi^{-1})^{(-1)})^T$ is a fundamental solution to (2a). By the obvious relation $\Phi[l]^\sharp=\Phi^\sharp[l]$, we can give the explicit formula of $\Phi[l]$.

Suppose $\Phi=(\phi',\phi)$, $\Phi^\sharp=\frac{1}{\det\Phi^{(-1)}}(K\phi^{(-1)},-K\phi'^{-1})$. Then the first column of $\Phi^\sharp[l]$ is given by

$$\Phi^\sharp[l]_1=\frac{1}{\det\Phi^{(-1)}}\begin{bmatrix}\dfrac{\det(\Lambda_{l+1}A_{l+1}(K\phi^{(-1)},z),A_{l+1}(h_1,\zeta_1),\ldots,A_{l+1}(h_l,\zeta_l))\prod\zeta_i}{\det(A_l(h_1,\zeta_1),\ldots,A_l(h_l,\zeta_l))}\\[2ex]\dfrac{\det(\Lambda_{l+1}B_{l+1}(K\phi^{(-1)},z),B_{l+1}(h_1,\zeta_1),\ldots,B_{l+1}(h_l,\zeta_l))}{\det(A_l(h_1,\zeta_1),\ldots,A_l(h_l,\zeta_l))}\end{bmatrix}.$$

Since $\Phi[l]^\sharp = \frac{((\Phi[l]^*)^T)^{(-1)}}{(\det\Phi[l])^{(-1)}} = \frac{(K\phi[l], -K\phi'[l])^{(-1)}}{(\det\mathcal{D}_2(l)^\perp)^{(-1)}\det\Phi^{(-1)}}$, equalling the first column we get (20d). Then (20e) and (20f) are obtained like (20c) and (20d). □

Remark 4. The l-iteration formulae for MDT 3 can be worked out analogously.

4. Generalized MDT of type 1

It is known that *positon* and *negaton*, see [35, 36], can be derived from Darboux transformation with double eigenvalues. We present the *generalized MDT 1* (GMDT 1) to deal with double or multiple eigenvalues. By this means the positon, negaton solutions for ALESCS can be obtained. Based on (18) the GMDT 1 is established by considering limit procedures during which the eigenvalues tend to be the same. We should mention that in (18e), the coefficient $\prod_{1\leqslant i\neq j\leqslant 2l}\left(\zeta_j^2-\zeta_i^2\right)^{-1}$ will be singular. To overcome this, the coefficient $\alpha_j(t)$ must be chosen carefully. Here we choose $\alpha_j(t) = \exp(\Omega_j b_j(t))$ where Ω_j is used to cancel this singularity.

Lemma 1. *Let $X_1 = X_1(\zeta)$ and $X_2 = X_2(\zeta)$ be n-dimensional vectors depending on parameter ζ analytically. Let $\zeta_i = \zeta + \epsilon\omega_i$, ($\omega_i \in \mathbb{C}, i = 1, \ldots, m, m \leqslant n$) be distinct constants. Let $b_i = b_i(t)$ be arbitrary functions of t. Let $\Omega_j = \frac{1}{(m-1)!}\prod_{1\leqslant i\neq j\leqslant m}(\zeta_j - \zeta_i)$. Then the determinant of an $n\times n$ matrix (We assume last $(n-m)$ columns are independent of ϵ and hence omit them.)*

$$\det(X_1(\zeta_1) + e^{\Omega_1 b_1}X_2(\zeta_1), \ldots, X_1(\zeta_m) + e^{\Omega_m b_m}X_2(\zeta_m), \ldots, \ldots, \ldots)$$

has the following leading term when $\epsilon \ll 1$

$$\frac{\prod_{1\leqslant i<j\leqslant m}(\omega_j-\omega_i)}{1!\cdots(m-1)!}\det\left(X, \partial_\zeta X, \ldots, \partial_\zeta^{m-2}X, \partial_\zeta^{m-1}X + \sum_{i=1}^m b_i X_2, \ldots, \ldots, \ldots\right)\epsilon^{(m-1)m/2},$$

where $X := X_1(\zeta) + X_2(\zeta)$.

Proof. By use the Taylor expansion of X at $\epsilon = 0$. □

Hereafter ζ_i $(i = 1, \ldots, p)$ will be simple eigenvalues, corresponding to eigenfunctions F_i, G_i of (12) (with $V = V_k, k = 1, 2, 3$). Let $H_i = F_i + \alpha_i(t)G_i$. Complex numbers ξ_i $(i = 1, \ldots, q)$ denote eigenvalues corresponding to eigenfunctions f_i and g_i. Let $h_i = f_i + g_i$. Denote β_i $(i = 1, \ldots, q)$ which are arbitrary functions that we will use in the following. We call an eigenvalue ξ_i has the multiplicity $m_i \geqslant 2$, if there are ω_{ij} $(j = 1, \ldots, m_i)$ and eigenvalues $\xi_{ij} = \xi_i + \epsilon\omega_{ij}$, such that ξ_{ij} are simple eigenvalues for eigenfunctions $f_{ij} = f_i(\xi_{ij})$ and $g_{ij} = g_i(\xi_{ij})$ of (12) when $\epsilon \neq 0$. Let $\mathbf{m} := (m_1, \ldots, m_q)$ be an array denoting the multiplicities. Note $|\mathbf{m}| = \sum_{j=1}^q m_j$. Suppose $p + |\mathbf{m}| = 2l$ for some integer l.

To be convenient, we define some notions. Let $A[p, \mathbf{m}]$ be

$$\begin{aligned}A[p,\mathbf{m}] = \det\big(&A_{2l}(H_1,\zeta_1), \ldots, A_{2l}(H_p,\zeta_p),\\ &A_{2l}(h_1,\xi_1), \partial_{\xi_1}A_{2l}(h_1,\xi_1), \ldots, \partial_{\xi_1}^{m_1-1}A_{2l}(h_1,\xi_1) + \beta_1 A_{2l}(g_1,\xi_1),\\ &\ldots,\ldots,\ldots,\ldots,\\ &A_{2l}(h_q,\xi_q), \partial_{\xi_q}A_{2l}(h_q,\xi_q), \ldots, \partial_{\xi_q}^{m_q-1}A_{2l}(h_q,\xi_q) + \beta_q A_{2l}(g_q,\xi_q)\big),\end{aligned}$$

$B[p, \mathbf{m}], \ldots, E[p, \mathbf{m}]$ is defined similarly by replacing letter A with $B, \ldots, E$. Let

$$\begin{aligned}X[p,\mathbf{m}](\psi) = \det\big(&X_{2l+1}(\psi,z), X_{2l+1}(H_1,\zeta_1), \ldots, X_{2l+1}(H_p,\zeta_p),\\ &X_{2l+1}(h_1,\xi_1), \partial_{\xi_1}X_{2l+1}(h_1,\xi_1), \ldots, \partial_{\xi_1}^{m_1-1}X_{2l+1}(h_1,\xi_1) + \beta_1 X_{2l+1}(g_1,\xi_1),\\ &\ldots,\ldots,\ldots,\ldots,\\ &X_{2l+1}(h_q,\xi_q), \partial_{\xi_q}X_{2l+1}(h_q,\xi_q), \ldots, \partial_{\xi_q}^{m_q-1}X_{2l+1}(h_q,\xi_q) + \beta_q X_{2l+1}(g_q,\xi_q)\big),\end{aligned}$$

$Y[p, \mathbf{m}](\psi)$ are defined similarly by replacing the letter X by Y.

Theorem 4 (GMDT 1)**.** *Recalled that H_i with ζ_i for $i = 1, \ldots, p$, h_i with ξ_i of multiplicity m_i for $i = 1, \ldots, q$, the following l-iteration of GMDT 1*

$$Q[p,\mathbf{m}] = -\frac{B[p,\mathbf{m}]^{(1)}}{A[p,\mathbf{m}]^{(1)}} - Q\frac{C[p,\mathbf{m}]^{(1)}}{A[p,\mathbf{m}]^{(1)}} \tag{21a}$$

$$R[p,\mathbf{m}] = -\frac{D[p,\mathbf{m}]^{(1)}}{A[p,\mathbf{m}]^{(1)}} - R\frac{E[p,\mathbf{m}]^{(1)}}{A[p,\mathbf{m}]^{(1)}} \tag{21b}$$

$$\psi_i[p,\mathbf{m}] = \left(\frac{X[p,\mathbf{m}](\psi_i)}{A[p,\mathbf{m}]}, z_i\frac{Y[p,\mathbf{m}](\psi_i)}{A[p,\mathbf{m}]}\right)^T, \qquad 1 \leqslant i \leqslant N \tag{21c}$$

$$\phi_i[p,\mathbf{m}] = \Xi_i\left(\frac{z_i Y[p,\mathbf{m}]\big(K\phi_i^{(-1)}\big)}{E[p,\mathbf{m}]}, -\frac{X[p,\mathbf{m}]\big(K\phi_i^{(-1)}\big)}{E[p,\mathbf{m}]}\right)^{(1)T} \tag{21d}$$

$$\psi_{N+s}[p,\mathbf{m}] = \left(\frac{X[p,\mathbf{m}](F_s)}{A[p,\mathbf{m}]}, \frac{\zeta_i Y[p,\mathbf{m}](F_s)}{A[p,\mathbf{m}]}\right)^T, \qquad 1 \leqslant s \leqslant p, \tag{21e}$$

$$\phi_{N+s}[p,\mathbf{m}] = -\frac{\Xi_{N+s}\partial_t \log\alpha_s}{\det(F_s, G_s)^{(1)}}\left(\frac{\zeta_s Y[p,\mathbf{m}](G_s)}{E[p,\mathbf{m}]}, -\frac{X[p,\mathbf{m}](G_s)}{E[p,\mathbf{m}]}\right)^{(1)T} \tag{21f}$$

$$\psi_{N+p+r}[p,\mathbf{m}] = \left(\frac{X[p,\mathbf{m}](f_r)}{A[p,\mathbf{m}]}, \frac{\xi_r Y[p,\mathbf{m}](f_r)}{A[p,\mathbf{m}]}\right)^T, \qquad 1 \leqslant r \leqslant q, \tag{21g}$$

$$\phi_{N+p+r}[p,\mathbf{m}] = -\frac{\Xi_{N+p+r}\dot{\beta}_r}{\det(f_r, g_r)^{(1)}}\left(\frac{\xi_r Y[p,\mathbf{m}](g_r)}{E[p,\mathbf{m}]}, -\frac{X[p,\mathbf{m}](g_r)}{E[p,\mathbf{m}]}\right)^{(1)T}, \tag{21h}$$

where

$$\Xi_i := \frac{z_i^{2l}}{\prod_{1\leqslant s\leqslant p}\left(z_i^2 - \zeta_s^2\right)\prod_{1\leqslant r\leqslant q}\left(z_i^2 - \xi_r^2\right)^{m_r}}$$

$$\Xi_{N+s} := \frac{\zeta_s^{2l-2}}{\prod_{1\leqslant j\neq s\leqslant p}\left(\zeta_s^2 - \zeta_j^2\right)\prod_{1\leqslant r\leqslant q}\left(\zeta_s^2 - \xi_r^2\right)^{m_r}}$$

$$\Xi_{N+p+r} := \frac{\xi_r^{2l-m_r-1}}{\prod_{1\leqslant j\leqslant p}\left(\xi_r^2 - \zeta_j^2\right)\prod_{1\leqslant j\neq r\leqslant q}\left(\xi_r^2 - \xi_j^2\right)^{m_j}2^{m_r-1}(m_r-1)!}$$

give a new solution to (11) with new sources $\psi_{N+s}[p,\mathbf{m}], \phi_{N+s}[p,\mathbf{m}]$ corresponding to $z_{N+s} = \zeta_s$ ($s = 1, \ldots, p$) and $\psi_{N+p+r}[p,\mathbf{m}], \phi_{N+p+r}[p,\mathbf{m}]$ corresponding to $z_{N+p+r} = \xi_r$ ($r = 1, \ldots, q$).

Proof. For l-iteration of MDT 1 (18), we have $2l$ eigenfunctions, including $H_i = F_i + \alpha_i G_i$ ($i = 1, \ldots, p$) w.r.t. ζ_i and $h_{i,j} = f_{i,j} + \mathrm{e}^{\Omega_{i,j}b_{i,j}(t)}g_{i,j}$ with respect to eigenvalues $\xi_{ij} = \xi_i + \epsilon_i\omega_{ij}$ ($i = 1, \ldots, q$, $j = 1, \ldots, m_i$). Let $\Omega_{i,j} = \frac{\prod_{1\leqslant k\neq j\leqslant m_i}(\xi_{i,j}-\xi_{i,k})}{(m_i-1)!}$ and $\beta_i = \sum_{1\leqslant j\leqslant m_i} b_{i,j}$. Let $\epsilon_i \to 0$ by using lemma 1. Note that sources terms obtained by the limit of (18e) w.r.t. ξ_i differ only by scalar multiplications. Thus summing up these terms we get (21h). □

5. Reduced GMDT 1 for D-NLS, D-mKdV with self-consistent sources

5.1. Reduction from ALESCS to D-NLSSCS and D-mKdVSCS

The Lax equations (12) and corresponding systems admit the following reductions Hereafter $J = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ and $\varepsilon = \pm 1$. With these constraints, the corresponding systems can be

Table 1. Reduction conditions.

Reduced system	D-NLSSCS	D-mKdVSCS
t-part matrix	$V = V_2$	$V = V_3$
Constraints on potential	$R = \varepsilon Q^*$	$R = \varepsilon Q$
The number of sources	$N = 2m$	$N = 2m$
Constraints on sources	$J\psi^*_{2j-1} = \psi_{2j}$	$J\psi_{2j-1} = \psi_{2j}$
	$J\phi^*_{2j-1} = -\phi_{2j}$	$J\phi_{2j-1} = -\phi_{2j}$
	$1/z^*_{2j-1} = z_{2j}$	$1/z_{2j-1} = z_{2j}$

Table 2. Constraints on eigenfunctions.

D-NLSSCS	D-mKdVSCS
$f_2 = Jf_1^*$	$f_2 = Jf_1$
$g_2 = Jg_1^*$	$g_2 = Jg_1$
$\alpha_2 = \alpha_1^*$	$\alpha_2 = \alpha_1$
$\zeta_2 = 1/\zeta_1^*$	$\zeta_2 = 1/\zeta_1$

- D-NLSSCS:

$$\mathrm{i}Q_t = (1-\varepsilon|Q|^2)\left(Q^{(1)} + Q^{(-1)} - \mathrm{i}\sum_{j=1}^{m}\left(\psi_j^1\phi_j^2 - \varepsilon\psi_j^{2*}\phi_j^{1*}\right)\right) - 2Q, \tag{22a}$$

$$\psi_j^{(1)} = U(z_j)\psi_j, \qquad \phi_j^{(-1)} = U(z_j)^T\phi_j, \qquad j = 1, \ldots, m, \tag{22b}$$

- D-mKdVSCS:

$$Q_t = (1-\varepsilon Q^2)\left(Q^{(1)} - Q^{(-1)} - \sum_{j=1}^{m}\left(\psi_j^1\phi_j^2 - \varepsilon\psi_j^2\phi_j^1\right)\right), \tag{23a}$$

$$\psi_j^{(1)} = U(z_j)\psi_j, \qquad \phi_j^{(-1)} = U(z_j)^T\phi_j, \qquad j = 1, \ldots, m. \tag{23b}$$

The Lax representations of D-NLSSCS and D-mKdVSCS are (12) with corresponding constraints.

5.2. Reduced MDT 1 for D-NLSSCS and D-mKdVSCS

Proposition 3. *(reduced MDT 1 for D-NLSSCS and D-mKdVSCS) Suppose f_1 and g_1 are eigenfunctions of D-NLSSCS' (D-mKdVSCS') Lax pair with eigenvalue ζ_1. Then f_2 and g_2 listed in table 2 are eigenfunctions of D-NLSSCS' (D-mKdVSCS') Lax pair with eigenvalue ζ_2. Let $h_i = f_i + \alpha_i g_i$ $(i = 1, 2)$, then theorem 1 gives new solutions to D-NLSSCS (D-mKdVSCS).*

Proof. The first statement can be verified by straightforward calculation. According to theorem 1, the 'DT'ed variables are solutions to the un-reduced system. It is sufficient to show that new variables satisfy the constraints in table 1. For instance, since $h_2 = Jh_1^*$, $\zeta_2 = 1/\zeta_1^*$, we have $\varepsilon\sigma^* = \tau$, $v^* = u$ and $u/v = \zeta_1^{*2}/\zeta_1^2$ in (17). Then conclusion can be drawn easily for D-NLSSCS. □

Remark 5. By virtue of Darboux transformation, it is easy to see that l-iteration formulae (18) of reduced MDT 1 for D-NLSSCS and D-mKdVSCS, $\{f_{2i-1}, g_{2i-1}, \alpha_{2i-1}, \zeta_{2i-1}\}$ and $\{f_{2i}, g_{2i}, \alpha_{2i}, \zeta_{2i}\}$ $(i = 1, \ldots, l)$ should satisfy constraints in table 2.

5.3. Reduced GMDT 1 for D-NLSSCS

The GMDT 1 is useful for degenerate solutions. However, The degenerate patterns are more complicated for reduced systems than the un-reduced systems. Let us discuss the degenerate pattern for D-NLSSCS first.

Assume $|\zeta| = 1$ and the fundamental solution for D-NLSSCS's Lax pair is defined by $\Phi(\zeta) = (f(\zeta), g(\zeta))$. Then $\Phi(\zeta)^\dagger := J\Phi(\zeta)^*$ is also a fundamental solution to D-NLSSCS's Lax pair with eigenvalue $\zeta^\dagger := 1/\zeta^* = \zeta$. Thus $\Phi(\zeta)^\dagger = \Phi(\zeta)P(t)$ must hold for some non-singular 2×2 matrix $P(t)$. If $P(t)$ has $\lambda(t)$ as its eigenvalue, then it is possible to construct $h(\zeta) = f(\zeta) + \alpha(t)g(\zeta)$ such that $h^\dagger$ and h are linear dependent, which is a case we must deal with in discussing the degeneration of multi-iteration of MDT 1 for D-NLSSCS. In fact, this is a very important case from which the positon solution is derived.

For simplicity, we only discuss m-iteration of MDT 1 with ζ an eigenvalue of multiplicity m. The general case can be found analogously. Suppose $f_j = f(\zeta_j), g_j = g(\zeta_j)$ are independent eigenfunctions for D-NLSSCS's Lax pair with eigenvalue $\zeta_j = \zeta + \epsilon\omega_j$. Let $f_j^\dagger = Jf_j^*, g_j^\dagger = Jg_j^*, h_j = f_j + \mathrm{e}^{\Omega_j b_j} g_j, h_j^\dagger = Jh_j^* = f_j^\dagger + \mathrm{e}^{\Omega_j^* b_j^*} g_j^\dagger, h := f + g, h^\dagger = f^\dagger + g^\dagger$. Then there are two different cases.

Case 1. $|\zeta| \neq 1$ or $|\zeta| = 1$ but $h^\dagger$ and h are linearly independent, in this case we can define $\Omega_i := \frac{\prod_{j\neq i}(\zeta_i - \zeta_j)}{(m-1)!}$. Thus, $\{h_j, \zeta_j\}$ and $\left\{h_j^\dagger, \zeta_j^\dagger\right\}$ fall into two different groups of multiplicities m. We call ζ and $\zeta^\dagger$ are both eigenvalues of multiplicities m.

Case 2. $|\zeta| = 1$ and $h(\zeta)^\dagger = \lambda(\zeta, t)h(\zeta)$. In this case, we must define $\Omega_i = \frac{\prod_{j\neq i}(\zeta_i - \zeta_j)\prod_{j=1}^m(\zeta_i - \zeta_j^\dagger)}{(2m-1)!}, h_j = f_j + \mathrm{e}^{\Omega_j b_j} g_j$. All $h_j, h_j^\dagger$ fall into one group of multiplicity $2m$ when $\epsilon \to 0$. This is a much complicated case, in which we deal with a sub-case, where $h(\zeta)^\dagger = \lambda h(\zeta^\dagger)$ for all $\zeta \in \mathbb{C}$ and λ does not depend on ζ. In this case, we call ζ is an eigenvalue of multiplicity $n = 2m$.

We have the following lemma (for e.g. we use $A_{2m}(h, \zeta)$).

Lemma 2. *The leading term of determinant*

$$\det\big(A_{2m}(f_1, \zeta_1) + \mathrm{e}^{\Omega_1 b_1} A_{2m}(g_1, \zeta_1), \ldots, A_{2m}(f_m, \zeta_m) + \mathrm{e}^{\Omega_m b_m} A_{2m}(g_m, \zeta_m),$$
$$A_{2m}\left(f_1^\dagger, \zeta_1^\dagger\right) + \mathrm{e}^{\Omega_1^* b_1^*} A_{2m}\left(g_1^\dagger, \zeta_1^\dagger\right), \ldots, A_{2m}\left(f_m^\dagger, \zeta_m^\dagger\right) + \mathrm{e}^{\Omega_m^* b_m^*} A_{2m}\left(g_m^\dagger, \zeta_m^\dagger\right)\big)$$

is for case 1:

$$c_1\epsilon^{m(m-1)} \det\big(A_{2m}(h, \zeta), \ldots, \partial_\zeta^{m-1} A_{2m}(h, \zeta) + \beta A_{2m}(g, \zeta),$$
$$A_{2m}(h^\dagger, \zeta^\dagger), \ldots, \partial_{\zeta^\dagger}^{m-1} A_{2m}(h^\dagger, \zeta^\dagger) + \beta^\dagger A_{2m}(g^\dagger, \zeta^\dagger)\big)$$

where

$$c_1 = \frac{\prod_{1\leqslant i<j\leqslant m}(\omega_{m+j} - \omega_{m+i})(\omega_j - \omega_i)}{(1! \cdots (m-1)!)^2}, \qquad \beta := \sum_{i=1}^m b_i, \beta^\dagger := (-1)^{m-1}\zeta^{*2m-2}\beta^*.$$

For case 2:

$$c_2\epsilon^{m(m-1)} \det\big(A_{2m}(h, \zeta), \partial_\zeta A_{2m}(h, \zeta), \ldots, \partial_\zeta^{2m-2} A_{2m}(h, \zeta),$$
$$\partial_\zeta^{2m-1} A_{2m}(h, \zeta) + \beta A_{2m}(g, \zeta) - \zeta^{-4m+2}\beta^* A_{2m}(g^\dagger/\lambda, \zeta)\big),$$

where

$$c_2 = \lambda^m \frac{\prod_{1\leqslant i<j\leqslant 2m}(\omega_j - \omega_i)}{1! \cdots (2m-1)!}.$$

Proof. For case 1, $\Omega_i^* b_i^* = \epsilon^{m-1}\frac{(-1)^{m-1}}{(m-1)!}\zeta^{*2m-2}b_i^*\prod_{j\neq i}(\omega_{m+i}-\omega_{m+j}) + o(\epsilon^{m-1})$ where we define $\zeta_i^\dagger - \zeta^\dagger = \epsilon\omega_{m+i}$. Let $b_i^\dagger := (-1)^{m-1}\zeta^{*2m-2}b_i^*$, $\beta^\dagger = \sum_{i=1}^m b_i^\dagger$, by lemma 1, we obtained the result.

For case 2, $\Omega_i = \frac{1}{(2m-1)!}\prod_{j\neq i}(\zeta_i-\zeta_j)\prod_{j=1}^m\left(\zeta_i-\zeta_j^\dagger\right)$, then $\Omega_i^* = -\frac{\epsilon^{2m-1}}{(2m-1)!}\prod_{j\neq i}(\omega_{m+i}-\omega_{m+j})\prod_{j=1}^m(\omega_{m+j}-\omega_j)\zeta^{-4m+2}$. Extracting λ from each of the last m columns, we obtain the result by using $2m$ in lemma 1. □

Suppose F_i and G_i are p pairs of eigenfunctions of D-NLSSCS' Lax pair with simple eigenvalue ζ_i. $H_i := F_i + \alpha_i(t)G_i$. Let f_j, g_j be q pairs of eigenfunctions of case 1 with eigenvalue ξ_j. Let $h_j = f_j + g_j$. Denote $\beta_j(t)$ to be q arbitrary functions that we shall use in the following. Let $\tilde{f}_k, \tilde{g}_k$ be r pairs of eigenfunctions with eigenvalue ω_k such that $\tilde{h}_j = \tilde{f}_j + \tilde{g}_j$ satisfies case 2. Let $\tilde{\beta}_k(t)$ be r arbitrary functions. Let $\mathbf{m} = (m_1,\ldots,m_q;n_1,\ldots,n_r)$ be an array denoting the multiplicities of cases 1 and 2. Define $|\mathbf{m}| := \sum_{j=1}^q m_j + \frac{1}{2}\sum_{k=1}^r n_k$, $p + |\mathbf{m}| = l$. To be convenient, we define the following determinants:

$$
\begin{aligned}
A^\sharp[p,\mathbf{m}] = \det\big(&A_{2l}(H_1,\zeta_1),\ldots,A_{2l}(H_p,\zeta_p),A_{2l}\big(H_1^\dagger,\zeta_1^\dagger\big),\ldots,A_{2l}\big(H_p^\dagger,\zeta_p^\dagger\big),\\
&A_{2l}(h_1,\xi_1),\partial_{\xi_1}A_{2l}(h_1,\xi_1),\ldots,\partial_{\xi_1}^{m_1-1}A_{2l}(h_1,\xi_1)+\beta_1A_{2l}(g_1,\xi_1),\\
&A_{2l}\big(h_1^\dagger,\xi_1^\dagger\big),\partial_{\xi_1^\dagger}A_{2l}\big(h_1^\dagger,\xi_1^\dagger\big),\ldots,\partial_{\xi_1^\dagger}^{m_1-1}A_{2l}\big(h_1^\dagger,\xi_1^\dagger\big)+\beta_1^\dagger A_{2l}\big(g_1^\dagger,\xi_1^\dagger\big),\\
&\ldots,\ldots,\ldots,\\
&A_{2l}(h_q,\xi_q),\partial_{\xi_q}A_{2l}(h_q,\xi_q),\ldots,\partial_{\xi_q}^{m_q-1}A_{2l}(h_q,\xi_q)+\beta_qA_{2l}(g_q,\xi_q),\\
&A_{2l}\big(h_q^\dagger,\xi_q^\dagger\big),\partial_{\xi_q^\dagger}A_{2l}\big(h_q^\dagger,\xi_q^\dagger\big),\ldots,\partial_{\xi_q^\dagger}^{m_q-1}A_{2l}\big(h_q^\dagger,\xi_q^\dagger\big)+\beta_q^\dagger A_{2l}\big(g_q^\dagger,\xi_q^\dagger\big),\\
&A_{2l}(\tilde{h}_1,\omega_1),\partial_{\omega_1}A_{2l}(\tilde{h}_1,\omega_1),\ldots,\partial_{\omega_1}^{2n_1-1}A_{2l}(\tilde{h}_1,\omega_1)+\tilde{\beta}_1A_{2l}(\tilde{g}_1,\omega_1)\\
&-\omega_1^{-4n_1+2}\tilde{\beta}_1^*A_{2l}(\tilde{g}_1/\lambda_1,\omega_1),\\
&\ldots,\ldots,\ldots,\\
&A_{2l}(\tilde{h}_r,\omega_r),\partial_{\omega_r}A_{2l}(\tilde{h}_r,\omega_r),\ldots,\partial_{\omega_r}^{2n_r-1}A_{2l}(\tilde{h}_r,\omega_r)+\tilde{\beta}_rA_{2l}(\tilde{g}_r,\omega_r)\\
&-\omega_r^{-4n_r+2}\tilde{\beta}_r^*A_{2l}\big(\tilde{g}_r^\dagger/\lambda_r,\omega_r\big)\big).
\end{aligned}
$$

And $B^\sharp[p,\mathbf{m}],\ldots,E^\sharp[p,\mathbf{m}]$ are defined similarly by replacing letter A with $B,\ldots,E$. Let

$$
\begin{aligned}
X^\sharp[p,\mathbf{m}](\psi) = \det\big(&X_{2l+1}(\psi,z),\\
&X_{2l+1}(H_1,\zeta_1),\ldots,X_{2l+1}(H_p,\zeta_p),X_{2l+1}\big(H_1^\dagger,\zeta_1^\dagger\big),\ldots,X_{2l+1}\big(H_p^\dagger,\zeta_p^\dagger\big),\\
&X_{2l+1}(h_1,\xi_1),\partial_{\xi_1}X_{2l+1}(h_1,\xi_1),\ldots,\partial_{\xi_1}^{m_1-1}X_{2l+1}(h_1,\xi_1)+\beta_1X_{2l+1}(g_1,\xi_1),\\
&X_{2l+1}\big(h_1^\dagger,\xi_1^\dagger\big),\partial_{\xi_1^\dagger}X_{2l+1}\big(h_1^\dagger,\xi_1^\dagger\big),\ldots,\partial_{\xi_1^\dagger}^{m_1-1}X_{2l+1}\big(h_1^\dagger,\xi_1^\dagger\big)+\beta_1^\dagger X_{2l+1}\big(g_1^\dagger,\xi_1^\dagger\big),\\
&\ldots,\ldots,\ldots,\\
&X_{2l+1}(h_q,\xi_q),\partial_{\xi_q}X_{2l+1}(h_q,\xi_q),\ldots,\partial_{\xi_q}^{m_q-1}X_{2l+1}(h_q,\xi_q)+\beta_qX_{2l+1}(g_q,\xi_q),\\
&X_{2l+1}\big(h_q^\dagger,\xi_q^\dagger\big),\partial_{\xi_q^\dagger}X_{2l+1}\big(h_q^\dagger,\xi_q^\dagger\big),\ldots,\partial_{\xi_q^\dagger}^{m_q-1}X_{2l+1}\big(h_q^\dagger,\xi_q^\dagger\big)+\beta_q^\dagger X_{2l+1}\big(g_q^\dagger,\xi_q^\dagger\big),\\
&X_{2l+1}(\tilde{h}_1,\omega_1),\partial_{\omega_1}X_{2l+1}(\tilde{h}_1,\omega_1),\ldots,\partial_{\omega_1}^{2n_1-1}X_{2l+1}(\tilde{h}_1,\omega_1)+\tilde{\beta}_1X_{2l+1}(\tilde{g}_1,\omega_1)\\
&-\omega_1^{-4n_1+2}\tilde{\beta}_1^*X_{2l+1}(\tilde{g}_1/\lambda_1,\omega_1),\\
&\ldots,\ldots,\ldots,\\
&X_{2l+1}(\tilde{h}_r,\omega_r),\partial_{\omega_r}X_{2l+1}(\tilde{h}_r,\omega_r),\ldots,\partial_{\omega_r}^{2n_r-1}X_{2l+1}(\tilde{h}_r,\omega_r)+\tilde{\beta}_rX_{2l+1}(\tilde{g}_r,\omega_r)\\
&-\omega_r^{-4n_r+2}\tilde{\beta}_r^*X_{2l+1}\big(\tilde{g}_r^\dagger/\lambda_r,\omega_r\big)\big)
\end{aligned}
$$

$Y^\sharp[p,\mathbf{m}](\psi)$ is defined similarly by replacing the letter X by Y.

Then reduced GMDT 1 for D-NLSSCS is defined as following

$$Q[p,\mathbf{m}] = -\frac{B^\sharp[p,\mathbf{m}]^{(1)}}{A^\sharp[p,\mathbf{m}]^{(1)}} - Q\frac{C^\sharp[p,\mathbf{m}]^{(1)}}{A^\sharp[p,\mathbf{m}]^{(1)}}, \tag{24a}$$

$$\psi_i[p,\mathbf{m}] = \left(\frac{X^\sharp[p,\mathbf{m}](\psi_i)}{A^\sharp[p,\mathbf{m}]}, z_i\frac{Y^\sharp[p,\mathbf{m}](\psi_i)}{A^\sharp[p,\mathbf{m}]}\right)^T, \qquad 1\leqslant i\leqslant N \tag{24b}$$

$$\phi_i[p,\mathbf{m}] = \Xi_i\cdot\left(\frac{z_iY^\sharp[p,\mathbf{m}]\big(K\phi_i^{(-1)}\big)}{E^\sharp[p,\mathbf{m}]}, -\frac{X^\sharp[p,\mathbf{m}]\big(K\phi_i^{(-1)}\big)}{E^\sharp[p,\mathbf{m}]}\right)^{(1)T}, \tag{24c}$$

$$\psi_{N+s}[p,\mathbf{m}] = \left(\frac{X^\sharp[p,\mathbf{m}](F_s)}{A^\sharp[p,\mathbf{m}]}, \frac{\zeta_iY^\sharp[p,\mathbf{m}](F_s)}{A^\sharp[p,\mathbf{m}]}\right)^T, \qquad 1\leqslant s\leqslant p, \tag{24d}$$

$$\phi_{N+s}[p,\mathbf{m}] = -\frac{\Xi_{N+s}\partial_t\log\alpha_s}{\det(F_s,G_s)^{(1)}}\left(\frac{\zeta_sY^\sharp[p,\mathbf{m}](G_s)}{E^\sharp[p,\mathbf{m}]}, -\frac{X^\sharp[p,\mathbf{m}](G_s)}{E^\sharp[p,\mathbf{m}]}\right)^{(1)T}, \tag{24e}$$

$$\psi_{N+p+s}[p,\mathbf{m}] = \left(\frac{X^\sharp[p,\mathbf{m}](f_s)}{A^\sharp[p,\mathbf{m}]}, \frac{\xi_sY^\sharp[p,\mathbf{m}](f_s)}{A^\sharp[p,\mathbf{m}]}\right)^T, \qquad 1\leqslant s\leqslant q, \tag{24f}$$

$$\phi_{N+p+s}[p,\mathbf{m}] = -\frac{\Xi_{N+p+s}\dot\beta_s}{\det(f_s,g_s)^{(1)}}\left(\frac{\xi_sY^\sharp[p,\mathbf{m}](g_s)}{E^\sharp[p,\mathbf{m}]}, -\frac{X^\sharp[p,\mathbf{m}](g_s)}{E^\sharp[p,\mathbf{m}]}\right)^{(1)T}, \tag{24g}$$

$$\psi_{N+p+q+s}[p,\mathbf{m}] = \left(\frac{X^\sharp[p,\mathbf{m}](\tilde f_s)}{A^\sharp[p,\mathbf{m}]}, \frac{\omega_sY^\sharp[p,\mathbf{m}](\tilde f_s)}{A^\sharp[p,\mathbf{m}]}\right)^T, \qquad 1\leqslant s\leqslant r, \tag{24h}$$

$$\phi_{N+p+q+s}[p,\mathbf{m}] = -\frac{\Xi_{N+p+q+s}\dot{\tilde\beta}_s}{\det(\tilde f_s,\tilde g_s)^{(1)}}\left(\frac{\omega_sY^\sharp[p,\mathbf{m}](\tilde g_s)}{E^\sharp[p,\mathbf{m}]}, -\frac{X^\sharp[p,\mathbf{m}](\tilde g_s)}{E^\sharp[p,\mathbf{m}]}\right)^{(1)T}, \tag{24i}$$

where the coefficients

$$\Xi_i = \frac{z_i^{2l}}{\prod_{1\leqslant s\leqslant p}\big(z_i^2-\zeta_s^2\big)\big(z_i^2-\zeta_s^{\dagger2}\big)\prod_{1\leqslant s\leqslant q}\big(z_i^2-\xi_s^2\big)^{m_s}\big(z_i^2-\xi_s^{\dagger2}\big)^{m_s}\prod_{1\leqslant s\leqslant r}\big(z_i^2-\omega_s^2\big)^{2n_r}},$$

$$\Xi_{N+s} = \frac{\zeta_s^{2l-2}\big/\big(\zeta_s^2-\zeta_s^{\dagger2}\big)}{\prod_{1\leqslant j\neq s\leqslant p}\big(\zeta_s^2-\zeta_j^2\big)\big(\zeta_s^2-\zeta_j^{\dagger2}\big)\prod_{1\leqslant j\leqslant q}\big(\zeta_s^2-\xi_j^2\big)^{m_j}\big(\zeta_s^2-\xi_j^{\dagger2}\big)^{m_j}\prod_{1\leqslant j\leqslant r}\big(\zeta_s^2-\omega_j^2\big)^{2n_j}},$$

$$\Xi_{N+p+s} = \frac{2^{1-m_s}\xi_s^{2l-m_s-1}\big/\big(\xi_s^2-\xi_s^{\dagger2}\big)^{m_s}}{\prod_{1\leqslant j\leqslant p}\big(\xi_s^2-\zeta_j^2\big)\big(\xi_s^2-\zeta_j^{\dagger2}\big)\prod_{1\leqslant j\neq s\leqslant q}\big(\xi_s^2-\xi_j^2\big)^{m_j}\big(\xi_s^2-\xi_j^{\dagger2}\big)^{m_j}\prod_{1\leqslant j\neq s\leqslant r}\big(\xi_s^2-\omega_j^2\big)^{2n_j}(m_s-1)!},$$

$$\Xi_{N+p+q+s} = \frac{2^{1-2n_s}\omega_s^{2l-2n_s-1}}{\prod_{1\leqslant j\leqslant p}\big(\omega_s^2-\zeta_j^2\big)\big(\omega_s^2-\zeta_j^{\dagger2}\big)\prod_{1\leqslant j\neq s\leqslant q}\big(\omega_s^2-\xi_j^2\big)^{m_j}\big(\omega_s^2-\xi_j^{\dagger2}\big)^{m_j}\prod_{1\leqslant j\neq s\leqslant r}\big(\omega_s^2-\omega_j^2\big)^{2n_j}(2n_s-1)!}.$$

The new sources $\psi_{N+s}[p,\mathbf{m}]$, $\phi_{N+s}[p,\mathbf{m}]$ correspond to $z_{N+s}=\zeta_s$ for $s=1,\ldots,p$, $\psi_{N+p+s}[p,\mathbf{m}]$, $\phi_{N+p+s}[p,\mathbf{m}]$ correspond to $z_{N+p+s}=\xi_s$ for $s=1,\ldots,q$, $\psi_{N+p+q+s}[p,\mathbf{m}]$, $\phi_{N+p+q+s}[p,\mathbf{m}]$ correspond to $z_{N+p+q+s}=\omega_s$ for $s=1,\ldots,r$, *5.4. Reduced GMDT 1 for D-mKdVSCS*

The degenerate pattern in this case is simpler than the former case. For simplicity, let $f_j = f(\zeta_j), g_j = g(\zeta_j)$ be independent eigenfunctions for D-mKdVSCS' Lax pair with eigenvalue $\zeta_j = \zeta + \epsilon\omega_j$. $f_j^{\ddagger} = Jf_j, g_j^{\ddagger} = Jg_j$. Let $h = f + g, h^{\ddagger} = f^{\ddagger} + g^{\ddagger}$. Then there is only one converge pattern.

- For $\zeta \neq \pm 1, h^{\ddagger}$ and h are linearly independent, in this case we can define $\Omega_i := \frac{\prod_{j\neq i}(\zeta_i-\zeta_j)}{(m-1)!}, h_j = f_j + e^{\Omega_j b_j} g_j$, then $\{h_j, \zeta_j\}$ and $\left\{h_j^{\ddagger}, \zeta_j^{\ddagger}\right\}$ fall into two different groups when $\epsilon \to 0$.

We have the following lemma (for e.g. we use $A_{2m}(h, \zeta)$).

Lemma 3. *The leading term of determinant*

$$\det\left(A_{2m}(f_1,\zeta_1) + e^{\Omega_1 b_1}A_{2m}(g_1,\zeta_1), \ldots, A_{2m}(f_m,\zeta_m) + e^{\Omega_m b_m}A_{2m}(g_m,\zeta_m),\right.$$
$$\left. A_{2m}\left(f_1^{\ddagger},\zeta_1^{\ddagger}\right) + e^{\Omega_1 b_1}A_{2m}\left(g_1^{\ddagger},\zeta_1^{\ddagger}\right), \ldots, A_{2m}\left(f_m^{\ddagger},\zeta_m^{\ddagger}\right) + e^{\Omega_m b_m}A_{2m}\left(g_m^{\ddagger},\zeta_m^{\ddagger}\right)\right)$$

is

$$c_3\epsilon^{m(m-1)}\det\left(A_{2m}(h,\zeta), \ldots, \partial_\zeta^{m-1}A_{2m}(h,\zeta) + \beta A_{2m}(g,\zeta),\right.$$
$$\left. A_{2m}(h^{\ddagger},\zeta^{\ddagger}), \ldots, \partial_{\zeta^{\ddagger}}^{m-1}A_{2m}(h^{\ddagger},\zeta^{\ddagger}) + \beta^{\ddagger}A_{2m}(g^{\ddagger},\zeta^{\ddagger})\right)$$

where $c_3 = \frac{\prod_{1\leqslant i<j\leqslant m}(\omega_{m+j}-\omega_{m+i})(\omega_j-\omega_i)}{(1!\cdots(m-1)!)^2}$, $\beta := \sum_{i=1}^m b_i, \beta^{\ddagger} := (-1)^{m-1}\zeta^{2m-2}\beta$.

Proof. For in this case, $\Omega_i b_i = \epsilon^{m-1}\frac{(-1)^{m-1}}{(m-1)!}\zeta^{2m-2}b_i\prod_{j\neq i}(\omega_{m+i} - \omega_{m+j}) + o(\epsilon^{m-1})$ where we define $\zeta_i^{\ddagger} - \zeta^{\ddagger} = \epsilon\omega_{m+i}$. Let $b_i^{\ddagger} := (-1)^{m-1}\zeta^{2m-2}b_i, \beta^{\ddagger} = \sum_{i=1}^m b_i^{\ddagger}$, by lemma 1, we have the result for this case. □

Suppose F_i and G_i are p pairs of eigenfunctions of D-mKdVSCS' Lax pair with simple eigenvalue ζ_i. $H_i := F_i + \alpha_i(t)G_i$ are linear combinations with arbitrary functions $\alpha_i(t)$. Let f_j, g_j be q pairs of eigenfunctions with eigenvalue $\xi_j \neq \pm 1$. Define $h_j = f_j + g_j$. $\beta_j(t)$ are q arbitrary functions that we shall use in the following. Let $\mathbf{m} = (m_1, \ldots, m_q)$ be an array denoting the multiplicities. Define $|\mathbf{m}| := \sum_{j=1}^q m_j$, then it is easy to see $p + |\mathbf{m}| = l$. We define the following determinants:

$$A^\flat[p,\mathbf{m}] = \det\left(A_{2l}(H_1,\zeta_1), \ldots, A_{2l}(H_p,\zeta_p), A_{2l}\left(H_1^{\ddagger},\zeta_1^{\ddagger}\right), \ldots, A_{2l}\left(H_p^{\ddagger},\zeta_p^{\ddagger}\right),\right.$$
$$A_{2l}(h_1,\xi_1), \partial_{\xi_1}A_{2l}(h_1,\xi_1), \ldots, \partial_{\xi_1}^{m_1-1}A_{2l}(h_1,\xi_1) + \beta_1 A_{2l}(g_1,\xi_1),$$
$$A_{2l}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right), \partial_{\xi_1^{\ddagger}}A_{2l}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right), \ldots, \partial_{\xi_1^{\ddagger}}^{m_1-1}A_{2l}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right) + \beta_1^{\ddagger}A_{2l}\left(g_1^{\ddagger},\xi_1^{\ddagger}\right),$$
$$\ldots,\ldots,\ldots,$$
$$A_{2l}(h_q,\xi_q), \partial_{\xi_q}A_{2l}(h_q,\xi_q), \ldots, \partial_{\xi_q}^{m_q-1}A_{2l}(h_q,\xi_q) + \beta_q A_{2l}(g_q,\xi_q),$$
$$\left. A_{2l}\left(h_q^{\ddagger},\xi_q^{\ddagger}\right), \partial_{\xi_q^{\ddagger}}A_{2l}\left(h_q^{\ddagger},\xi_q^{\ddagger}\right), \ldots, \partial_{\xi_q^{\ddagger}}^{m_q-1}A_{2l}\left(h_q^{\ddagger},\xi_q^{\ddagger}\right) + \beta_q^{\ddagger}A_{2l}\left(g_q^{\ddagger},\xi_q^{\ddagger}\right),\right),$$

and $B^\flat[p,\mathbf{m}], \ldots, E^\flat[p,\mathbf{m}]$ are defined similarly by replacing letter A with $B, \ldots, E$. Let

$$X^\flat[p,\mathbf{m}](\psi) = \det\left(X_{2l+1}(\psi, z),\right.$$
$$X_{2l+1}(H_1,\zeta_1), \ldots, X_{2l+1}(H_p,\zeta_p), X_{2l+1}\left(H_1^{\ddagger},\zeta_1^{\ddagger}\right), \ldots, X_{2l+1}\left(H_p^{\ddagger},\zeta_p^{\ddagger}\right),$$
$$X_{2l+1}(h_1,\xi_1), \partial_{\xi_1}X_{2l+1}(h_1,\xi_1), \ldots, \partial_{\xi_1}^{m_1-1}X_{2l+1}(h_1,\xi_1) + \beta_1 X_{2l+1}(g_1,\xi_1),$$
$$X_{2l+1}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right), \partial_{\xi_1^{\ddagger}}X_{2l+1}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right), \ldots, \partial_{\xi_1^{\ddagger}}^{m_1-1}X_{2l+1}\left(h_1^{\ddagger},\xi_1^{\ddagger}\right) + \beta_1^{\ddagger}X_{2l+1}\left(g_1^{\ddagger},\xi_1^{\ddagger}\right),$$

$$\ldots, \ldots, \ldots, \ldots,$$
$$X_{2l+1}(h_q, \xi_q), \partial_{\xi_q} X_{2l+1}(h_q, \xi_q), \ldots, \partial_{\xi_q}^{m_q-1} X_{2l+1}(h_q, \xi_q) + \beta_q X_{2l+1}(g_q, \xi_q),$$
$$X_{2l+1}\left(h_q^{\ddagger}, \xi_q^{\ddagger}\right), \partial_{\xi_q^{\ddagger}} X_{2l+1}\left(h_q^{\ddagger}, \xi_q^{\ddagger}\right), \ldots, \partial_{\xi_q^{\ddagger}}^{m_q-1} X_{2l+1}\left(h_q^{\ddagger}, \xi_q^{\ddagger}\right) + \beta_q^{\ddagger} X_{2l+1}\left(g_q^{\ddagger}, \xi_q^{\ddagger}\right),)$$

$Y^{\flat}[p, \mathbf{m}](\psi)$ is defined similarly by replacing the letter X by Y.

Then reduced GMDT 1 for D-mKdVSCS is defined as following

$$Q[p, \mathbf{m}] = -\frac{B^{\flat}[p, \mathbf{m}]^{(1)}}{A^{\flat}[p, \mathbf{m}]^{(1)}} - Q\frac{C^{\flat}[p, \mathbf{m}]^{(1)}}{A^{\flat}[p, \mathbf{m}]^{(1)}}, \tag{25a}$$

$$\psi_i[p, \mathbf{m}] = \left(\frac{X^{\flat}[p, \mathbf{m}](\psi_i)}{A^{\flat}[p, \mathbf{m}]}, z_i\frac{Y^{\flat}[p, \mathbf{m}](\psi_i)}{A^{\flat}[p, \mathbf{m}]}\right)^T, \qquad 1 \leqslant i \leqslant N, \tag{25b}$$

$$\phi_i[p, \mathbf{m}] = \Xi_i\left(\frac{z_i Y^{\flat}[p, \mathbf{m}]\left(K\phi_i^{(-1)}\right)}{E^{\flat}[p, \mathbf{m}]}, -\frac{X^{\flat}[p, \mathbf{m}]\left(K\phi_i^{(-1)}\right)}{E^{\flat}[p, \mathbf{m}]}\right)^{(1)T} \tag{25c}$$

$$\psi_{N+s}[p, \mathbf{m}] = \left(\frac{X^{\flat}[p, \mathbf{m}](F_s)}{A^{\flat}[p, \mathbf{m}]}, \frac{\zeta_i Y^{\flat}[p, \mathbf{m}](F_s)}{A^{\flat}[p, \mathbf{m}]}\right)^T, \qquad 1 \leqslant s \leqslant p, \tag{25d}$$

$$\phi_{N+s}[p, \mathbf{m}] = -\frac{\Xi_{N+s}\partial_t \log \alpha_s}{\det(F_s, G_s)^{(1)}}\left(\frac{\zeta_s Y^{\flat}[p, \mathbf{m}](G_s)}{E^{\flat}[p, \mathbf{m}]}, -\frac{X^{\flat}[p, \mathbf{m}](G_s)}{E^{\flat}[p, \mathbf{m}]}\right)^{(1)T} \tag{25e}$$

$$\psi_{N+p+s}[p, \mathbf{m}] = \left(\frac{X^{\flat}[p, \mathbf{m}](f_s)}{A^{\flat}[p, \mathbf{m}]}, \frac{\xi_s Y^{\flat}[p, \mathbf{m}](f_s)}{A^{\flat}[p, \mathbf{m}]}\right)^T, \qquad 1 \leqslant s \leqslant q, \tag{25f}$$

$$\phi_{N+p+s}[p, \mathbf{m}] = -\frac{\Xi_{N+p+s}\dot{\beta}_s}{\det(f_s, g_s)^{(1)}}\left(\frac{\xi_s Y^{\flat}[p, \mathbf{m}](g_s)}{E^{\flat}[p, \mathbf{m}]}, -\frac{X^{\flat}[p, \mathbf{m}](g_s)}{E^{\flat}[p, \mathbf{m}]}\right)^{(1)T}, \tag{25g}$$

where

$$\Xi_i = \frac{z_i^{2l}}{\prod_{1\leqslant s\leqslant p}\left(z_i^2 - \zeta_s^2\right)\left(z_i^2 - \zeta_s^{\ddagger 2}\right)\prod_{1\leqslant s\leqslant q}\left(z_i^2 - \xi_s^2\right)^{m_s}\left(z_i^2 - \xi_s^{\ddagger 2}\right)^{m_s}},$$
$$\Xi_{N+s} = \frac{\zeta_s^{2l-2}\big/\left(\zeta_s^2 - \zeta_s^{\ddagger 2}\right)}{\prod_{1\leqslant j\neq s\leqslant p}\left(\zeta_s^2 - \zeta_j^2\right)\left(\zeta_s^2 - \zeta_j^{\ddagger 2}\right)\prod_{1\leqslant j\leqslant q}\left(\zeta_s^2 - \xi_j^2\right)^{m_j}\left(\zeta_s^2 - \xi_j^{\ddagger 2}\right)^{m_j}},$$
$$\Xi_{N+p+s} = \frac{\xi_s^{2l-m_s-1}\left(\xi_s^2 - \xi_s^{\ddagger 2}\right)^{-m_s}2^{1-m_s}\big/(m_s-1)!}{\prod_{1\leqslant j\leqslant p}\left(\xi_s^2 - \zeta_j^2\right)\left(\xi_s^2 - \zeta_j^{\ddagger 2}\right)\prod_{1\leqslant j\neq s\leqslant q}\left(\xi_s^2 - \xi_j^2\right)^{m_j}\left(\xi_s^2 - \xi_j^{\ddagger 2}\right)^{m_j}}.$$

The new sources $\psi_{N+s}[p, \mathbf{m}]$, $\phi_{N+s}[p, \mathbf{m}]$ correspond to $z_{N+s} = \zeta_s$ for $s = 1, \ldots, p$, $\psi_{N+p+s}[p, \mathbf{m}]$, $\phi_{N+p+s}[p, \mathbf{m}]$ correspond to $z_{N+p+s} = \xi_s$ for $s = 1, \ldots, q$.

6. Solutions to ALFNFSCS, D-NLSSCS and D-mKdVSCS

The MDT and (reduced) GMDT techniques enable us to construct many types of solutions to ALFNFSCS, D-NLSSCS and D-mKdVSCS. Starting from trivial solutions of these systems, by choosing specific solutions to the Lax pairs and specific $p, \mathbf{m}$, we can construct multi-soliton solutions (both with vanishing or non-vanishing boundary condition), (multi)-positon, (multi)-negaton etc.

6.1. Solutions obtained by non-degenerate MDT of types 1 and 2

We start from a trivial solution $Q = R = 0$ with $N = 0$ (without source), which is a common solution for ALFNFSCS, D-NLSSCS and D-mKdVSCS. The Lax pair (12) becomes

$$\psi^{(1)} = \mathrm{diag}(z, z^{-1})\psi, \qquad \psi_t = \mathrm{diag}(\delta(z), -\delta(z))\psi, \tag{26}$$

where

$$\delta(z) = \begin{cases} z^2 & (\mathrm{ALFNFSCS}) \\ \mathrm{i}\left(1 - \frac{1}{2}(z^2 + z^{-2})\right) & (\mathrm{D-NLSSCS}) \\ \frac{1}{2}(z^2 + z^{-2}) & (\mathrm{D-mKdVSCS}). \end{cases} \tag{27}$$

$\delta(z)$ is called *dispersion relation* in [30]. A fundamental solution to (26) is

$$\Psi = (f, g) = \begin{pmatrix} z^n \, \mathrm{e}^{\delta t} & 0 \\ 0 & z^{-n} \, \mathrm{e}^{-\delta t} \end{pmatrix}.$$

Suppose $\zeta_i \in \mathbb{C}, i = 1, \ldots, 2l$ are eigenvalues, $\alpha_i(t)$ are $2l$ arbitrary smooth functions. Let $F_i := f(\zeta_i), G_i := g(\zeta_i)$.

$$H_i = F_i + \alpha_i G_i = \left(\zeta_i^n \, \mathrm{e}^{\delta(\zeta_i)t}, \alpha_i \zeta_i^{-n} \, \mathrm{e}^{-\delta(\zeta_i)t}\right)^T. \tag{28}$$

Then by using MDT 1 (18*a*)–(18*e*) or equivalently by using GMDT 1 (21*a*)–(21*f*) with $p = 2l$, taking the ALFNF dispersion relation, the l-soliton solution for ALFNF with $\leqslant 2l$ sources (depending on choices of $\alpha_j(t)$) can be constructed.

The more important cases are multi-soliton solutions to D-NLSSCS and D-mKdVSCS.

6.1.1. Multi-soliton solutions to D-NLSSCS and its interactions. Suppose $\zeta_i \in \mathbb{C}, i = 1, \ldots, l$ are distinct eigenvalues in $\{z \in \mathbb{C} | |z| < 1\}$. By letting $\delta_i := \delta(\zeta_i) = \mathrm{i}\left(1 - \frac{1}{2}\left(\zeta_i^2 + \zeta_i^{-2}\right)\right)$ in (28), using (24*a*)–(24*e*) with $p = l$ we get the l-soliton solution.

For $\varepsilon = -1$, the 1-soliton with two self-consistent sources is

$$\begin{aligned} Q[1] &= -\mathrm{e}^{\mathrm{i}\,\mathrm{Im}(Z+2\kappa)} \sinh(2\,\mathrm{Re}\,\kappa)\,\mathrm{sech}(\mathrm{Re}\,Z), \\ \psi_1[1] &= (\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\,Z - X + \tau}, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,Z + X})^T \sinh(2\,\mathrm{Re}\,\kappa)\,\mathrm{sech}(\mathrm{Re}(Z - 2\kappa)), \\ \phi_1[1] &= \frac{\dot{\tau}}{2}\left(\mathrm{e}^{X+\kappa-\tau-\mathrm{i}\,\mathrm{Im}(Z+2\kappa)}, -\mathrm{e}^{-X-\kappa+\mathrm{i}\,\mathrm{Im}(Z+2\kappa)}\right)^T \mathrm{sech}(\mathrm{Re}(Z + 2\kappa)), \\ \psi_2[1] &= \psi_1[1]^\dagger, \qquad \phi_2[1] = -\phi_1[1]^\dagger. \end{aligned}$$

where we recall the notation $\psi_1[1]^\dagger := J\psi_1[1]^*$. The symbols $\kappa = \log(\zeta_1)$, $\tau = \log(\alpha_1)$, $X = n\kappa + \delta_1 t$, $Z = 2X + \kappa - \tau$.

If set $\tau = 2\,\mathrm{Re}\,\delta_1 t$, then $\mathrm{Re}\,Z$ does not depend on t. Thus we got a non-travelling bounded solution with periodically changing of its amplitude, we call this a *breather* type solution.

For $\varepsilon = 1$, the 1-soliton with two self-consistent sources:

$$\begin{aligned} Q[1] &= \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(Z+2\kappa)} \sinh(2\,\mathrm{Re}\,\kappa)\,\mathrm{csch}(\mathrm{Re}\,Z), \\ \psi_1[1] &= (-\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\,Z - X + \tau}, \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,Z + X})^T \sinh(2\,\mathrm{Re}\,\kappa)\,\mathrm{csch}(\mathrm{Re}(Z - 2\kappa)), \\ \phi_1[1] &= \frac{\dot{\tau}}{2}\left(\mathrm{e}^{X+\kappa-\tau-\mathrm{i}\,\mathrm{Im}(Z+2\kappa)}, \mathrm{e}^{-X-\kappa+\mathrm{i}\,\mathrm{Im}(Z+2\kappa)}\right)^T \mathrm{csch}(\mathrm{Re}(Z + 2\kappa)), \\ \psi_2[1] &= \psi_1[1]^\dagger, \qquad \phi_2[1] = -\phi_1[1]^\dagger. \end{aligned}$$

This is a solution with singularities depending on zeros of $\mathrm{Re}\,Z$.

For $p = l$, we got the l-soliton solution where the potential

$$Q[l] = -\frac{\sum_{1 \leqslant i_1 < \cdots < i_{l-1} \leqslant 2l} b_{i_1,\ldots,i_{l-1}} \exp\left(-\sum_{j=1}^{l-1} \tilde{Z}_{i_j}\right)}{\sum_{1 \leqslant i_1 < \cdots < i_l \leqslant 2l} a_{i_1,\ldots,i_l} \exp\left(-\sum_{j=1}^{l} \tilde{Z}_{i_j}\right)}, \tag{29}$$

where if we set $j_1 < \cdots < j_{l+k}$ are indexes such that $\{j_1, \ldots, j_{l+k}\} = \{1, \ldots, 2l\} - \{i_1, \ldots, i_{l-k}\}$. Then

$$b_{i_1,\ldots,i_{l-1}} = (-1)^{\sum_{s=1}^{l+1}(s+j_s)} \varepsilon^{\sum_{s=1}^{l-1}(1+i_s)} V(j_1, \ldots, j_{l+1}) V(i_1, \ldots, i_{l-1}) \prod_{s=1}^{l-1} \tilde{\zeta}_{i_s}^2,$$

$$a_{i_1,\ldots,i_l} = (-1)^{\sum_{s=1}^{l}(s+j_s)} \varepsilon^{\sum_{s=1}^{l}(1+i_s)} V(j_1, \ldots, j_l) V(i_1, \ldots, i_l),$$

$$\tilde{\zeta}_i = \begin{cases} \zeta_m & i = 2m-1 \\ \zeta_m^\dagger & i = 2m, \end{cases} \qquad \tilde{Z}_i = \begin{cases} Z_m & i = 2m-1 \\ -Z_m^* & i = 2m, \end{cases} \qquad V(i_1, \ldots, i_m) := \det\left(\tilde{\zeta}_{i_s}^{2t-2}\right)_{s,t},$$

$\kappa_m = \log(\zeta_m)$, $\tau_m := \log \alpha_m$, $Z_m := (2n+1)\kappa_m + 2\delta_m t - \tau_m$.

For $\varepsilon = -1$, we can analyse the interactions of solitons in (29). For convenience, we assume $|\tau_i| < \infty$ as $|t| \to \infty$, and $0 < \mathrm{Re}\,\delta_1 < \mathrm{Re}\,\delta_2 < \cdots < \mathrm{Re}\,\delta_l$. Then for $i < j$

$$\mathrm{Re}\, Z_j - \mathrm{Re}\, Z_i = (1+2n)\,\mathrm{Re}(\kappa_j - \kappa_i) - \mathrm{Re}(\tau_j - \tau_i) + 2\,\mathrm{Re}(\delta_j - \delta_i)t,$$

which implies $\mathrm{Re}\, Z_j - \mathrm{Re}\, Z_i \to \pm\infty$ as $t \to \pm\infty$. Thus it is easy to see if we fix Z_k

$$\begin{aligned} Q[l] &\sim -\frac{b_{2,4,\ldots,2k-2,2k+1,2k+3,\ldots,2l-1}}{a_{2,4,\ldots,2k-2,2k-1,2k+1\cdots,2l-1}\exp(-\tilde{Z}_{2k-1}) + a_{2,4,\ldots,2k-2,2k,2k+1,\ldots,2l-1}\exp(-\tilde{Z}_{2k})} \\ &= \beta_k^- \exp(\mathrm{i}\,\mathrm{Im}\, Z_k)\,\mathrm{sech}(\mathrm{Re}\, Z_k - \log|\alpha_k^-| - \mathrm{i}\arg\alpha_k^-) \qquad (t \to -\infty), \\ Q[l] &\sim -\frac{b_{1,3,\ldots,2k-3,2k+2,2k+4,\ldots,2l}}{a_{1,3,\ldots,2k-3,2k-1,2k+2\cdots,2l}\exp(-\tilde{Z}_{2k-1}) + a_{1,3,\ldots,2k-3,2k,2k+2,\ldots,2l}\exp(-\tilde{Z}_{2k})} \\ &= \beta_k^+ \exp(\mathrm{i}\,\mathrm{Im}\, Z_k)\,\mathrm{sech}(\mathrm{Re}\, Z_k + \log|\alpha_k^+| + \mathrm{i}\arg\alpha_k^+) \qquad (t \to +\infty). \end{aligned}$$

where

$$\beta_k^- = (-1)^{l+1} \frac{\zeta_k^{\dagger 2} - \zeta_k^2}{2} \prod_{j=1}^{k-1} \zeta_j^{\dagger 2} \sqrt{\frac{\left(\zeta_k^2 - \zeta_j^2\right)\left(\zeta_k^{\dagger 2} - \zeta_j^2\right)}{\left(\zeta_k^2 - \zeta_j^{\dagger 2}\right)\left(\zeta_k^{\dagger 2} - \zeta_j^{\dagger 2}\right)}} \prod_{j=k+1}^{l} \zeta_j^2 \sqrt{\frac{\left(\zeta_j^{\dagger 2} - \zeta_k^2\right)\left(\zeta_j^{\dagger 2} - \zeta_k^{\dagger 2}\right)}{\left(\zeta_j^2 - \zeta_k^2\right)\left(\zeta_j^2 - \zeta_k^{\dagger 2}\right)}},$$

$$\alpha_k^- = \sqrt{\prod_{j=1}^{k-1} \frac{\left(\zeta_k^2 - \zeta_j^{\dagger 2}\right)\left(\zeta_k^{\dagger 2} - \zeta_j^2\right)}{\left(\zeta_k^{\dagger 2} - \zeta_j^{\dagger 2}\right)\left(\zeta_k^2 - \zeta_j^2\right)} \prod_{j=k+1}^{l} \frac{\left(\zeta_j^2 - \zeta_k^2\right)\left(\zeta_j^{\dagger 2} - \zeta_k^{\dagger 2}\right)}{\left(\zeta_j^2 - \zeta_k^{\dagger 2}\right)\left(\zeta_j^{\dagger 2} - \zeta_k^2\right)}}.$$

And $\alpha_k^+ = 1/\alpha_k^-$, $\beta_k^+ = \frac{(\zeta_k^{\dagger 2} - \zeta_k^2)^2}{4\beta_k^-} \prod_{j=1}^{k-1} \frac{\zeta_j^2}{\zeta_j^{*2}} \prod_{j=k+1}^{l} \frac{\zeta_j^2}{\zeta_j^{*2}}$. Then the profile has its centre shifted by $-2\frac{\log|\alpha_k^-| + \mathrm{i}\arg(\alpha_k^-)}{\log|\zeta_k|}$, amplitude enlarged by $\left|\frac{\beta_k^+}{\beta_k^-}\right|$, the phase shifted by $\arg\beta_k^+ - \arg\beta_k^-$.

6.1.2. Multi-soliton solutions to D-mKdVSCS and its interactions. Suppose $\zeta_i \in \mathbb{R}$, $|\zeta_i| < 1$ are l distinct eigenvalues. By letting $\delta_i := \delta(\zeta_i) = \frac{1}{2}\left(\zeta_i^2 - \zeta_i^{-2}\right)$ in (28), using (25*a*)–(25*e*) with $p = l$, we get l-soliton solution. For $\varepsilon = -1$, 1-soliton with 2 self-consistent sources is

$$\begin{aligned} Q[1] &= -\sinh(2\kappa)\,\mathrm{sech}(Z), \\ \psi_1[1] &= (\mathrm{e}^{\tau - X}, \mathrm{e}^{X})^T \sinh(2\kappa)\,\mathrm{sech}(Z - 2\kappa), \\ \phi_1[1] &= \frac{\dot{\tau}}{2}(\mathrm{e}^{X+\kappa-\tau}, -\mathrm{e}^{-X-\kappa})^T\,\mathrm{sech}(Z + 2\kappa), \\ \psi_2[1] &= \psi_1[1]^\ddagger, \qquad \phi_2[1] = \phi_1[1]^\ddagger, \end{aligned}$$

where we recall $\psi_1[1]^\ddagger := J\psi_1[1]$. For $\varepsilon = 1$, solution with two self-consistent sources is

$$Q[1] = \sinh(2\kappa)\,\mathrm{csch}(Z),$$
$$\psi_1[1] = (-\mathrm{e}^{\tau-X}, \mathrm{e}^{X})^T \sinh(2\kappa)\,\mathrm{csch}(Z-2\kappa),$$
$$\phi_1[1] = \frac{\dot\tau}{2}(\mathrm{e}^{X+\kappa-\tau}, \mathrm{e}^{-X-\kappa})^T\,\mathrm{csch}(Z+2\kappa),$$
$$\psi_2[1] = \psi_1[1]^{\ddagger}, \qquad \phi_2[1] = \phi_1[1]^{\ddagger}.$$

For l-soliton, analogue to the previous section, we can give the explicit formulae and analyse the interactions of soliton. Here if we assume $\zeta_i \in (0,1)$, distinct and τ_i are bounded functions for all t and $0 < \delta_1 < \cdots < \delta_l$, fix Z_k, then the profile has its centre shifted by $-2\frac{\log\alpha_k^-}{\log\zeta_k}$ and the amplitude enlarged by the scale $\frac{\beta_k^+}{\beta_k^-}$, where

$$\alpha_k^- = \sqrt{\prod_{j=1}^{k-1}\frac{\left(\zeta_k^{\ddagger 2}-\zeta_j^2\right)\left(\zeta_k^2-\zeta_j^{\ddagger 2}\right)}{\left(\zeta_k^2-\zeta_j^2\right)\left(\zeta_k^{\ddagger 2}-\zeta_j^{\ddagger 2}\right)}\prod_{j=k+1}^{l}\frac{\left(\zeta_j^{\ddagger 2}-\zeta_k^{\ddagger 2}\right)\left(\zeta_j^2-\zeta_k^2\right)}{\left(\zeta_j^{\ddagger 2}-\zeta_k^2\right)\left(\zeta_j^2-\zeta_k^{\ddagger 2}\right)}},$$

$$\beta_k^- = (-1)^{l+1}\sinh(2\log\zeta_k)\prod_{j=1}^{k-1}\zeta_j^{\ddagger 2}\sqrt{\frac{\left(\zeta_k^2-\zeta_j^2\right)\left(\zeta_k^{\ddagger 2}-\zeta_j^2\right)}{\left(\zeta_k^2-\zeta_j^{\ddagger 2}\right)\left(\zeta_k^{\ddagger 2}-\zeta_j^{\ddagger 2}\right)}}\prod_{j=k+1}^{l}\zeta_j^2\sqrt{\frac{\left(\zeta_j^{\ddagger 2}-\zeta_k^2\right)\left(\zeta_j^{\ddagger 2}-\zeta_k^{\ddagger 2}\right)}{\left(\zeta_l^2-\zeta_k^2\right)\left(\zeta_j^2-\zeta_k^{\ddagger 2}\right)}}.$$

And $\alpha_k^+ = 1/\alpha_k^-$, $\beta_k^+ = \sinh^2(2\log\zeta_k)/\beta_k^-$.

6.1.3. Solitons with non-vanishing boundary conditions for ALFNFSCS. MDT 2 can be used to construct solitons with non-vanishing boundary conditions. For example, if we start with a nonzero solution to (13), say $Q = q\,\mathrm{e}^{2(1-qr)t}$, $R = r\,\mathrm{e}^{-2(1-qr)t}$, $N = 0$ (q and r are constants). Then Lax pair for ALFNFSCS becomes

$$\psi^{(1)} = \begin{pmatrix} z & q\,\mathrm{e}^{2(1-qr)t} \\ r\,\mathrm{e}^{-2(1-qr)t} & 1/z \end{pmatrix}\psi, \qquad \psi_t = \begin{pmatrix} z^2-2qr & 2zq\,\mathrm{e}^{2(1-qr)t} \\ 2zr\,\mathrm{e}^{-2(1-qr)t} & -z^2 \end{pmatrix}\psi. \qquad (30)$$

The fundamental solution to (30) is

$$\Phi := (f, g) = \begin{pmatrix} (\kappa_+ - z^{-1})\kappa_+^n\,\mathrm{e}^{-\chi_1 t} & q\kappa_-^n\,\mathrm{e}^{-\chi_2 t+\delta t} \\ r\kappa_+^n\,\mathrm{e}^{-\chi_1 t-\delta t} & (\kappa_- - z)\kappa_-^n\,\mathrm{e}^{-\chi_2 t} \end{pmatrix}$$

for $\kappa_\pm = \frac{z+z^{-1}}{2} \pm \sqrt{\frac{(z-z^{-1})^2}{4}+qr}$, $\delta = 2(1-qr)$, $\chi_1 = -z^2+2z\kappa_- - \delta$, $\chi_2 = z^2 - 2z\kappa_- + 2$. By taking $z=\zeta$, $h = f + \alpha(t)g$ for $\alpha(t) = \exp(-\mathrm{i}\,\mathrm{Im}(\chi_1-\chi_2)t)$, using 1-iteration of MDT 2 (20a), (20b), (20e), (20f), we get the 1-soliton with non-vanishing boundary condition with one source for the ALFNFSCS.

$$Q[1] = (1-\kappa_-\zeta)q\,\mathrm{e}^{\delta t}\frac{1-\frac{\kappa_- -\zeta}{\kappa_- -\zeta^{-1}}A\,\mathrm{e}^{Z}}{1+A\,\mathrm{e}^{Z}},$$
$$R[1] = \frac{1}{\zeta}A\,\mathrm{e}^{-\delta t}\frac{1+B\,\mathrm{e}^{Z}}{1+A\,\mathrm{e}^{Z}},$$
$$\psi[1] = \left(1, \frac{\mathrm{e}^{-\delta t}}{\zeta q}\right)^T\frac{(\zeta\kappa_+ + \zeta^{-1}\kappa_- - \delta)\kappa_+^n\,\mathrm{e}^{-\chi_1 t}}{1+A\,\mathrm{e}^{Z^{(-1)}}},$$
$$\phi[1] = \frac{\dot\alpha}{\alpha^2}(q^{-1}\,\mathrm{e}^{-\delta t}, -\zeta)^T\frac{\kappa_-^{-n-2}\,\mathrm{e}^{\chi_2 t}}{1+A\,\mathrm{e}^{Z^{(1)}}},$$

where $A = \frac{\kappa_+ - \zeta^{-1}}{q}$, $B = \frac{r}{\kappa_- - \zeta}$, $Z := (n+1)\log\frac{\kappa_+}{\kappa_-} + (\chi_2-\chi_1-\delta)t - \log\alpha$.

The 2-soliton solution is constructed quite analogously by taking two different eigenvalue ζ_1 and ζ_2. We omitted it.

Although the MDT 2 is effective in construct solutions with nonzero boundary to the ALFNFSCS and also the other non-reduced equations in the ALHSCS, the question whether MDT 2 can be reduced to Darboux transformations for D-NLSSCS, D-mKdVSCS or not remains open.

6.2. *Solutions obtained by degenerate MDT 1—positons, negatons and pole solutions*

With the aid of (reduced) GMDT 1, we can construct many types of degenerate solutions of ALFNFSCS, D-NLSSCS and D-mKdVSCS. Since the GMDT 1 for ALFNFSCS is much simpler than the reduced GMDT 1 for D-NLSSCS and D-mKdVSCS, we restrict our discussion on solutions of D-NLSSCS and D-mKdVSCS.

6.2.1. D-NLSSCS case. Hereafter we will start with trivial solution $Q = R = 0$. We recall that the Lax pair for D-NLSSCS with a trivial solution is (26) with the dispersion relation, which can be found in (27).

Suppose $\xi \in \{|z| > 1\}$ is an eigenvalue, $\xi^\dagger$ is another eigenvalue. We set multiplicities of ξ and $\xi^\dagger$ by $\mathbf{m} = (2, 2;)$. Let $h = f + g = (\xi^n\,\mathrm{e}^{\delta t}, \xi^{-n}\,\mathrm{e}^{-\delta t})$, then $h^\dagger = Jh^*$ is an eigenfunction corresponding to $\xi^\dagger$. Let $\beta(t)$ be arbitrary functions of t, then by (24a), (24f) and (24g), we find the following solutions

$$Q[0, (2, 2;)] = 4\varepsilon\,\mathrm{e}^{2I+\mathrm{i}5\theta}\frac{\operatorname{Re} W\cosh^\varepsilon Z + (\mathrm{i}\operatorname{Im} W - 2\coth(2\kappa))\sinh^\varepsilon Z}{4\sinh^{-2}(2\kappa)(\sinh^\varepsilon(Z))^2 - \varepsilon|W|^2},$$

$$\psi_1[0, (2, 2;)] = 4\begin{pmatrix} \dfrac{\mathrm{e}^{-Z+\kappa} + \varepsilon\sinh(2\kappa)(W^* - \coth(2\kappa) - 1)\,\mathrm{e}^{Z-\kappa}}{4\sinh^{-2}(2\kappa)(\sinh^\varepsilon(Z-2\kappa))^2 - \varepsilon|W-2|^2}\,\mathrm{e}^{-\frac{Z-\kappa}{2}+I+2\mathrm{i}\theta} \\ \dfrac{\varepsilon\sinh(2\kappa)(W^* + \coth(2\kappa) - 1)\,\mathrm{e}^{-Z+\kappa} - \mathrm{e}^{Z-\kappa}}{4\sinh^{-2}(2\kappa)(\sinh^\varepsilon(Z-2\kappa))^2 - \varepsilon|W-2|^2}\,\mathrm{e}^{\frac{Z-\kappa}{2}-I-2\mathrm{i}\theta} \end{pmatrix},$$

$$\phi_1[0, (2, 2;)] = -\frac{\dot\beta}{2}\begin{pmatrix} \dfrac{\varepsilon\sinh(2\kappa)\,\mathrm{e}^{-Z-\kappa}(W^* + 1 - \coth(2\kappa)) - \mathrm{e}^{Z+\kappa}}{4(\sinh^\varepsilon(Z+2\kappa))^2 - \varepsilon|W+2|^2\sinh^2(2\kappa)}\,\mathrm{e}^{\frac{Z-\kappa}{2}-I-2\mathrm{i}\theta} \\ -\dfrac{\varepsilon\sinh(2\kappa)(W^* + 1 - \coth(2\kappa))\,\mathrm{e}^{Z+\kappa} + \mathrm{e}^{-Z-\kappa}}{4(\sinh^\varepsilon(Z+2\kappa))^2 - \varepsilon|W+2|^2\sinh^2(2\kappa)}\,\mathrm{e}^{\frac{-Z+\kappa}{2}+I+4\mathrm{i}\theta} \end{pmatrix},$$

$$\psi_2[0, (2, 2;)] = \psi_1[0, (2;)]^\dagger \qquad \phi_2[0, (2;)] = -\phi_1[0, (2;)]^\dagger,$$

where $W = 2n + 1 - 2\mathrm{i}(\xi^2 - \xi^{-2}) - \xi\beta$, $Z = (2n+1)\kappa + 2\operatorname{Re}\delta t$, $I = in\theta + \mathrm{i}\operatorname{Im}\delta t$, $\kappa = \log|\xi|$, $\theta = \arg\xi$, and

$$\sinh^\varepsilon(z) := \begin{cases} \sinh(z) & \text{for} \quad \varepsilon = 1 \\ \cosh(z) & \text{for} \quad \varepsilon = -1, \end{cases} \qquad \cosh^\varepsilon(z) := \begin{cases} \cosh(z) & \text{for} \quad \varepsilon = 1 \\ \sinh(z) & \text{for} \quad \varepsilon = -1. \end{cases}$$

It is easy to see if $\varepsilon = 1$ then $Q[0, (2, 2;)]$ is a function decreasing exponentially as $|n| \to \infty$. The singularities of which appear at zeros of $4\sinh^\varepsilon(Z)^2 - \sinh^2(2\kappa)|W|^2$. We call such a solution the *negaton* solution for D-NLSSCS equation. While if $\varepsilon = -1$ then $Q[0, (2, 2;)]$ is also exponentially decreasing when $|n| \to \infty$. However in this case the denominator of $Q[0, (2, 2;)]$ is always positive, so in this case the solution has no singularities. We call this solution a *pole* solution for D-NLSSCS equation.

Next we assume $\varepsilon = 1$. Let $\omega_i \in \{|z| = 1\}$ for $i = 1, 2$ be two eigenvalues. The degeneracies for these eigenvalue are $\mathbf{m} = (; 2, 2)$. The corresponding eigenfunction are $\tilde h_i = \left(\omega_i^n\,\mathrm{e}^{\delta_i t}, \omega_i^{-n}\,\mathrm{e}^{-\delta_i t}\right)^T$. The corresponding arbitrary functions are $\tilde\beta_i$. For simplicity, we assume $\tilde{\beta}_1 \in \mathbb{R}$, $\tilde{\beta}_2 = 0$. By (24*a*), (24*h*) and (24*i*), we find the following solution

$$Q[0,(;2,2)] = 2\mathrm{e}^{2\mathrm{i}(\theta_1+\theta_2)} \frac{\mathrm{e}^{\mathrm{i}Z_1}[W_2 - 2\mathrm{i}\cot(\theta_2-\theta_1)] + \mathrm{e}^{\mathrm{i}Z_2}[W_1 - 2\tilde{\beta}_1\cos\theta_1 - 2\mathrm{i}\cot(\theta_1-\theta_2)]}{4\sin^{-2}(\theta_1-\theta_2)\sin^2(\frac{1}{2}(Z_1-Z_2)) - (W_1 - 2\tilde{\beta}_1\cos\theta_1)W_2},$$

$$\psi_1[0,(;2,2)] = 4\begin{pmatrix} \dfrac{\mathrm{e}^{\mathrm{i}(Z_2-\theta_2)}+\mathrm{e}^{\mathrm{i}(Z_1-\theta_1)}[i(W_2-1)\sin(\theta_1-\theta_2)-\cos(\theta_1-\theta_2)]}{4\sin^2(\frac{1}{2}(Z_1-Z_2)^{(-1)})\sin^{-2}(\theta_1-\theta_2)-(W_1^{(-1)}-2\tilde{\beta}_1\cos\theta_1)W_2^{(-1)}}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)-\frac{i}{2}(Z_1-\theta_1)} \\ \dfrac{[i(W_2-1)\sin(\theta_1-\theta_2)+\cos(\theta_1-\theta_2)]\,\mathrm{e}^{-\mathrm{i}(Z_1-\theta_1)}-\mathrm{e}^{-\mathrm{i}(Z_2-\theta_2)}}{4\sin^2(\frac{1}{2}(Z_1-Z_2)^{(-1)})\sin^{-2}(\theta_1-\theta_2)-(W_1^{(-1)}-2\tilde{\beta}_1\cos\theta_1)W_2^{(-1)}}\,\mathrm{e}^{\frac{i}{2}(Z_1-\theta_1)-i(\theta_1+\theta_2)} \end{pmatrix},$$

$$\phi_1[0,(;2,2)] = -\frac{\dot{\tilde{\beta}}_1}{2}\begin{pmatrix} -\dfrac{\mathrm{e}^{-\mathrm{i}(Z_2-\theta_2)+2\mathrm{i}(\theta_1-\theta_2)}+[(W_2+1)\sin(\theta_2-\theta_1)i-\cos(\theta_1-\theta_2)]\,\mathrm{e}^{-\mathrm{i}(Z_1-\theta_1)}}{4\sin^2(\frac{1}{2}(Z_1-Z_2)^{(1)})-\sin^2(\theta_1-\theta_2)(W_1^{(1)}-2\tilde{\beta}_1\cos\theta_1)W_2^{(1)}}\,\mathrm{e}^{\frac{i}{2}(Z_1-\theta_1)-i(\theta_1-\theta_2)} \\ \dfrac{\mathrm{e}^{\mathrm{i}(Z_2-\theta_2)+2\mathrm{i}(\theta_2-\theta_1)}+[(W_2-1)\sin(\theta_1-\theta_2)i-\cos(\theta_1-\theta_2)]\,\mathrm{e}^{\mathrm{i}(Z_1-\theta_1)}}{4\sin^2(\frac{1}{2}(Z_1-Z_2)^{(1)})-\sin^2(\theta_1-\theta_2)(W_1^{(1)}-2\tilde{\beta}_1\cos\theta_1)W_2^{(1)}}\,\mathrm{e}^{-\frac{i}{2}(Z_1-\theta_1)+i(3\theta_1+\theta_2)} \end{pmatrix},$$

$$\psi_2[0,(;2,2)] = \psi_1[0,(;2,2)]^\dagger, \qquad \phi_2[0,(;2,2)] = -\phi_1[0,(;2,2)]^\dagger,$$

where $\theta_i = \arg\omega_i$, $Z_i = (2n+1)\theta_i + 4\sin^2\theta_i t$, $W_i = 2n+1+4\sin(2\theta_i)t$. It is not hard to see that $|Q[0,(;2,2)]|$ decays to 0 in a speed $O(|n|)$ as $|n| \to \infty$. It oscillates since there are the term $\mathrm{e}^{\mathrm{i}Z_j}$ for $j = 1, 2$. It has singularities at zeros of $4\sin^2(\frac{1}{2}(Z_1-Z_2)) - \sin^2(\theta_1-\theta_2)(W_1 - 2\tilde{\beta}_1\cos\theta_1)W_2$. For such reason, we call this a *positon* solution to the D-NLSSCS.

6.2.2. D-mKdVSCS case. We recall that the Lax pair for D-mKdVSCS with a trivial solution is (26) with the dispersion relation, which can be found in (27).

Suppose $\xi > 1$ is a real eigenvalue, $\xi^\ddagger := \xi^{-1}$ is another eigenvalue. We set the multiplicities of ξ and $\xi^\ddagger$ by $\mathbf{m} = (2, 2;)$. Then analogous to the D-NLSSCS case, let $h = f + g = (\xi^n \mathrm{e}^{\delta t}, \xi^{-n}\mathrm{e}^{-\delta t})^T$, $h^\ddagger$ is an eigenfunction corresponding to ξ^{-1}, β_1 is arbitrary functions of t, then by (25*a*), (25*f*) and (25*g*), we find the following solutions

$$Q[0,(2,2)] = 4\varepsilon\frac{(W+1)\cosh^\varepsilon(Z+\kappa) - 2\coth(2\kappa)\sinh^\varepsilon(Z+\kappa)}{4(\sinh^\varepsilon(Z+\kappa))^2\sinh^{-2}(2\kappa) - \varepsilon(W+1)^2},$$

$$\psi_1[0,(2,2)] = 4\begin{pmatrix} \dfrac{\mathrm{e}^{-Z} + \varepsilon\sinh(2\kappa)(W - \coth(2\kappa))\,\mathrm{e}^{Z}}{4(\sinh^\varepsilon(Z-\kappa))^2\sinh^{-2}(2\kappa) - \varepsilon(W-1)^2}\,\mathrm{e}^{-Z/2} \\ -\dfrac{e^{Z} - \varepsilon\sinh(2\kappa)(W + \coth(2\kappa))\,\mathrm{e}^{-Z}}{4(\sinh^\varepsilon(Z-\kappa))^2\sinh^{-2}(2\kappa) - \varepsilon(W-1)^2}\,\mathrm{e}^{Z/2} \end{pmatrix},$$

$$\phi_1[0,(2,2)] = \frac{\dot{\beta}_1}{2}\begin{pmatrix} \dfrac{\mathrm{e}^{Z+2\kappa} - \varepsilon\sinh(2\kappa)(W + 2 + \coth(2\kappa))\,\mathrm{e}^{-Z-2\kappa}}{4(\sinh^\varepsilon(Z+3\kappa))^2 - \varepsilon\sinh^2(2\kappa)(W+3)^2}\,\mathrm{e}^{Z/2+2\kappa} \\ \dfrac{\mathrm{e}^{-Z-2\kappa} + \varepsilon\sinh(2\kappa)(W + 2 - \coth(2\kappa))\,\mathrm{e}^{Z+2\kappa}}{4(\sinh^\varepsilon(Z+3\kappa))^2 - \varepsilon\sinh^2(2\kappa)(W+3)^2}\,\mathrm{e}^{-Z/2} \end{pmatrix},$$

$$\psi_2[0,(2,2)] = \psi_1[0,(2,2)]^\ddagger, \qquad \phi_2[0,(2,2)] = -\phi_1[0,(2,2)]^\ddagger,$$

where $\kappa = \log\xi$, $Z = 2n\kappa + 2\sinh(2\kappa)t$, $W = 2n + 4\cosh(2\kappa)t - e^\kappa\beta_1$.

It is easy to see if $\varepsilon = 1$ then $Q[0,(2,2)]$ is a function fast decay and possess singularity (determined by zeros of $4\sinh^\varepsilon(Z+\kappa)^2\sinh^{-2}(2\kappa) - \varepsilon(W+1)^2$). We call such a solution the *negaton* solution for D-mKdVSCS equation. While if $\varepsilon = -1$ then $Q[0,(2,2)]$ is also exponentially decreasing when $|n| \to \infty$. However in this case the denominator of $Q[0,(2,2)]$ is always positive, so the solution has no singularities. We call this solution a *pole* solution for the D-mKdVSCS equation.

Now suppose $\omega \in \{|z| = 1\}$ is an eigenvalue, $\omega^\ddagger$ is another eigenvalue, the multiplicities of ω and $\omega^\ddagger$ are indicated by $\mathbf{m} = (2, 2)$. Then analogous to the D-NLSSCS case, let $h = f + g = (\omega^n \mathrm{e}^{\delta t}, \omega^{-n} \mathrm{e}^{-\delta t})$, $h^\ddagger$ is an eigenfunction corresponding to ω^{-1}, $\beta_1 = a(t)/\omega$, $\beta_2 = -\omega a(t)$ where $a(t)$ is arbitrary functions of t. Then by (25a), (25f) and (25g), we find the following solutions

$$Q[0, (2, 2)] = 4\sqrt{\varepsilon}\frac{(2W + 1 - a)\cos^\varepsilon(2Z + \theta) - 2\cot(2\theta)\sin^\varepsilon(2Z + \theta)}{4\sin^{-2}(2\theta)\sin^{\varepsilon 2}(2Z + \theta) - (2W + 1 - a)^2},$$

$$\psi_1[0, (2, 2)] = 4\varepsilon\begin{pmatrix} \dfrac{\mathrm{e}^{-2\mathrm{i}Z} + \varepsilon\mathrm{i}\sin(2\theta)(2W - a + \mathrm{i}\cot(2\theta))\,\mathrm{e}^{2\mathrm{i}Z}}{4\sin^{\varepsilon 2}(2Z - \theta)\sin^{-2}(2\theta) - (2W - 1 - a)^2}\,\mathrm{e}^{-\mathrm{i}Z} \\ \dfrac{\varepsilon\mathrm{i}\sin(2\theta)(2W - a - \mathrm{i}\cot(2\theta))\,\mathrm{e}^{-2\mathrm{i}Z} - \mathrm{e}^{2\mathrm{i}Z}}{4\sin^{\varepsilon 2}(2Z - \theta)\sin^{-2}(2\theta) - (2W - 1 - a)^2}\,\mathrm{e}^{\mathrm{i}Z} \end{pmatrix},$$

$$\phi_1[0, (2, 2)] = -\frac{\varepsilon\dot{a}}{2\sin^2(2\theta)}\begin{pmatrix} \dfrac{\mathrm{e}^{2\mathrm{i}Z+2\mathrm{i}\theta} - \varepsilon\mathrm{i}\sin(2\theta)\,\mathrm{e}^{-2\mathrm{i}Z-2\mathrm{i}\theta}(2W + 2 - a - \mathrm{i}\cot(2\theta))}{4\sin^{\varepsilon 2}(2Z + 3\theta)\sin^{-2}(2\theta) - (2W + 3 - a)^2}\,\mathrm{e}^{\mathrm{i}Z+\mathrm{i}\theta} \\ \dfrac{\mathrm{e}^{-2\mathrm{i}\theta-2\mathrm{i}Z} + \varepsilon\mathrm{i}\sin(2\theta)(2W + 2 - a + \mathrm{i}\cot(2\theta))\,\mathrm{e}^{2\mathrm{i}Z+2\mathrm{i}\theta}}{4\sin^{\varepsilon 2}(2Z + 3\theta)\sin^{-2}(2\theta) - (2W + 3 - a)^2} \end{pmatrix},$$

$$\psi_2[0, (2, 2)] = \psi_1[0, (2, 2)]^\ddagger, \qquad \phi_2[0, (2, 2)] = -\phi_1[0, (2, 2)]^\ddagger,$$

where $\theta = \arg\omega$, $Z = n\theta + \sin(2\theta)t$, $W = n + 2\cos(2\theta)t$,

$$\sin^\varepsilon(z) := \begin{cases} \sin(z) & \text{for} \quad \varepsilon = 1 \\ \cos(z) & \text{for} \quad \varepsilon = -1, \end{cases} \qquad \cos^\varepsilon(z) := \begin{cases} \cos(z) & \text{for} \quad \varepsilon = 1 \\ \sin(z) & \text{for} \quad \varepsilon = -1. \end{cases}$$

They are slowly decaying, oscillating and singular solutions. We call it *positons* for D-mKdVSCS.

7. Conclusions and problems

In this paper, a systematic study of Darboux transformations and their applications to ALESCSs and reduced systems have been given. It turns out that 'modified' Darboux transformations, which differ from original DT by permitting linear combinations of eigenfunction up to arbitrary functions of t, are more straightforward than binary DT with arbitrary functions of t, for the construction of explicit solutions to SESCSs. The 'variation of constant' is important both in MDT and binary DT to solve equations with sources. It turns out such an idea is also important in constructing new SESCSs. In paper [28], where Hu and his coworkers varied coefficients in Grammian-type solutions by arbitrary functions of t to construct bilinear forms of SESCSs.

This paper also deals with the construction of various types of solutions and discussed primarily some analytic properties for this solutions. Confidently solutions of combined type such as soliton–positon, soliton–negaton and positon–negaton and even higher order positons and negatons can be constructed through GMDT 1 by suitable choosing of $\mathbf{m}$ and specific eigenvalues. However, these topics are not included in our present paper. And the dark solitons for D-NLSSCS and D-mKdVSCS, which are also very important topics, are not investigated too. It can also be a question whether the MDT 2 and 3 can be applied to reduced systems of ALESCS or not. We hope we can discuss these problems in the future.

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